FLUCTUATIONS IN ENERGY LOST BY HIGH ENERGY CHARGED PARTICLES IN PASSING THROUGH MATTER

A Thesis

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by

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CHAPTER I

INTRODUCTION

1. The Problem.

The determination of the distribution in energy loss by particles traversing a given thickness of matter can be separated into two problems: 1) the determination of the cross sections for the collisions by which the particle may lose energy; 2) the calculation of the distribution in total energy loss resulting from such collisions. The first problem has been treated by many authors (9)*,(10),(11),(12),(13),(14),(15),(16),(17), and the cross sections given by the present relativistic quantum theory for collisions with atoms, of electrons, positrons, protons, and mesons of spin 0, ½, or 1, may be regarded as known to a sufficient approximation. The second problem may be formulated as follows: given the probability $\chi(E')dE'dt$ that in going a distance dt, a particle will suffer a collision in which it loses an amount of energy between E' and $E'+dE'$, to calculate the probability $P(\epsilon,t)d\epsilon$ that after traversing a thickness t the particle has lost a total amount of energy between ϵ and $\epsilon+d\epsilon$. The development of general methods of attacking this problem, and their application to the calculation of $P(\epsilon,t)$ for high energy mesons traversing various thicknesses of materials, constitute the central purpose of this thesis.

Methods of determining the distribution $P(\epsilon,t)$ will be described which are applicable to problems of this type whenever the function $\chi(E')$ is known. The methods are particularly suited to cases when $\chi(E')$ is such that the probability that a particle will lose a large fraction of its kinetic energy in a single collision is small, as is the case for collisions of heavy particles with electrons. These methods will be applied to the determination of $P(\mathcal{E},t)$ for mesons of cosmic ray energies. Curves will be presented which can be used in the analysis of experiments involving meson energies from 1 to 1000 Mev, and any thicknesses of (homogeneous) material. The curves are also applicable to protons or other charged particles of mass large compared with the electron mass and of kinetic energies from 0.01 to 10 times the rest energy of the particles. Only energy losses due to collisions with electrons have been taken into account in the calculations.

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* Numbers in parentheses refer to the Bibliographical Notes.

The problem of the effect of scattering by collisions on the spatial distribution and the distribution in direction of motion will not be treated in this thesis. The angular deflections suffered by mesons are very small except at very low energies and except for rare close collisions with an atomic nucleus. When directional scattering is important, the distribution $P(\mathcal{E},t)$ can still be determined by the methods to be described, but the variable t must be interpreted as the actual path length of the particle in the material. The problem of directional scattering has been studied by many authors (4),(5),(6),(7). The effect of scattering on the path length t can be treated approximately by methods similar to those described by Rossi and Greisen (4,p.268), or by Rose (7).

2. Historical Review.

The problem of determining the distribution in energy loss of fast electrons was probably first attacked by Bohr (1) in 1915. Bohr derived, using classical theory and the Rutherford-Bohr model of the atom, an expression for $\chi(E')$ for electrons and α-particles. $\chi(E')$ turned out to be proportional to $1/E'^2$ except for very small (or very large) values of E', a result which also follows from the quantum mechanical treatment (4). Bohr also derived formulas for the mean total energy loss $\bar{\mathcal{E}}$, including relativistic corrections, and for the mean square fluctuation in energy loss $\overline{(\Delta\mathcal{E})^2}$, in terms of $\chi(E')$ and the thickness of material t; his formulas are valid when $\bar{\mathcal{E}}$ is small compared with the initial kinetic energy. From these he derived correct formulas for the mean range and mean square fluctuation in range in terms of $\chi(E')$. For α-particles, Bohr was able to show that, except for extremely small thicknesses, the energy loss distribution is nearly Gaussian and is therefore given by*:

$$P(\mathcal{E}) = \frac{1}{\left(2\pi\overline{(\Delta\mathcal{E})^2}\right)^{1/2}} e^{-\frac{(\mathcal{E}-\bar{\mathcal{E}})^2}{2\overline{(\Delta\mathcal{E})^2}}} \qquad 2.1$$

This result depends essentially on the fact that the energy loss E' in any one collision of an α-particle with an electron is very small. For β-particles, which can lose a large fraction of their energy in a single collision, it is necessary to modify the treatment by omitting at first all collisions in which an energy greater than E'_r is lost, where E'_r is so chosen that the average number of collisions in which $E'>E'_T$ is very small, while collisions in which $E'<E'_T$ occur very frequently. The energy loss due to collisions in which $E'<E'_T$ can then be shown to have approximately a Gaussian distribution:

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* The notations throughout are mine.

$$P_{(E'<E'_T)}(\mathcal{E}) = \frac{1}{\left(2\pi\overline{(\Delta\mathcal{E})^2_T}\right)^{1/2}} e^{-\frac{(\mathcal{E}-\bar{\mathcal{E}}_T)^2}{2\overline{(\Delta\mathcal{E})^2_T}}} \qquad 2.2$$

$\bar{\mathcal{E}}_T$ will depend only very slightly on the precise value of E'_T if E'_T is correctly chosen. It was shown that collisions in which $E'>E'_T$ have little effect on the position of the peak of the curve $P(\mathcal{E})$, and presumably merely add a small tail in the direction of larger values of $\mathcal{E}$. Bohr compared his numerical results with the meager data available at that time and showed that the agreement was satisfactory.

Bohr's treatment of electron straggling was extended and improved in 1929 by Williams (2). Williams's method was to choose E'_T so that on the average the electron suffers one collision for which $E'>E'_T$ in traversing the thickness t of matter. He showed that then $P_{(E'<E'_T)}(\mathcal{E},t)$ is correctly given by 2.2. The distribution of total energy loss due to collisions in which $E'>E'_T$ can be represented by the first few terms in a series the nth term of which represents the distribution of energy loss for those electrons which suffer just n collisions. Combining these two distributions according to the rules of probability theory, treating the two groups of energy losses as independent random variables, Williams was able to calculate numerically the function $P(\mathcal{E})$, where $\mathcal{E}$ is the total energy loss due to all collisions. It turns out that $P(\mathcal{E},t)$ may be expressed in terms of a single universal function ϕ as follows:

$$P(E,t) = \frac{1}{\xi}\,\phi\left(\frac{E-E_p}{\xi}\right) \qquad 2.3$$

where

$$\xi = At$$

$$E_p = \xi\left(\log\frac{\xi}{\epsilon} + j\right) \qquad 2.4$$

Here A and ϵ are constants depending on the kind of material traversed, and j is a universal constant whose value according to Williams is 0.30. ξ has the dimensions of an energy; A is [energy / length]; ϵ is an energy.* The function ϕ can be determined from a table given by Williams (2,p.445). A comparison with experimental results for the straggling of 150 kv to 260 kv electrons in traversing thin foils resulted in good agreement as to the shape of the curve $P(E,t)$, but there was apparently a discrepancy of a factor of 2.3 in the value of A. Williams attributed this to a failure of the classical theory for collisions in which E' is of the order of 1000 volts.

Williams has written a number of later papers comparing experimental and theoretical results for the stopping of α- and β-particles (8),(9). In (8), he compares the function $\chi(E')$ determined from the quantum theoretical calculations of Gaunt (32) with Bohr's classical expression. The two functions agree for values of E' large compared with the atomic energy levels. The average rate of energy loss $\int_0^\infty E'\chi(E')\,dE'$ is also very nearly the same for both theories. But according to the quantum theory the function $\chi(E')$ for small E' is mainly concentrated in and except for extremely thin foils where ξ (equation 2.4 above) is comparable with the electronic ionization potentials, (e.g., less than about 0.005 cm of lead for $\beta^2 = 0.9$). The assumption that $\chi(E')$ is proportional to $1/E'^2$ for large values of E' is valid for values of E' small compared with the maximum energy E'_m that can be lost by the incident particle in a single collision with a free electron; the result is that Landau's results are valid for small thicknesses of material. (The condition is $\xi \ll E'_m$, i. e. for mesons of 10 Mev, $t \ll 0.3$ cm of lead. The condition becomes less restrictive at higher energies. For electrons the condition becomes $\xi \ll \frac{1}{2} E_k$). Landau observed that, if we neglect the dependence of $\chi(E')$ on the energy of the ionizing particle, the problem of determining $P(\varepsilon, t)$ can be solved rigorously by a Laplace transformation on the variable ε. Under the assumptions stated, he showed that $P(\varepsilon, t)$ is expressible in the form 2.3, where ξ and ε_p are given by 2.4, and he gave an explicit expression for the Laplace transform of ϕ. The inversion of this transform was carried out numerically, and curves for ϕ and $\int_{(\varepsilon-\varepsilon_p)/\xi}^{\infty} \phi\left(\frac{\varepsilon-\varepsilon_p}{\xi}\right) d\left(\frac{\varepsilon-\varepsilon_p}{\xi}\right)$ are given in Landau's paper.*

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* The notation in 2.3 and 2.4 is due to Landau ([illegible]).

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* There is a discrepancy in the curves as given, in that, for $(\varepsilon-\varepsilon_p)/\xi$ between about 0 and 6, one is not the integral of the other. The curve for ϕ was checked at one point by numerical integration and it appears to be correct; the integral curve is then too low by about 0.025, as read on the vertical scale, for $(\varepsilon-\varepsilon_p)/\xi$ between 1 and 4. The corrected Landau curves are those marked "$\lambda_w = 1.48$" in Fig. 1a and 1b at the end of this thesis.

Landau's ϕ curve should agree with Williams's results since both are solutions of the same mathematical problem. Actually, the tails of the two curves agree for $\frac{(\epsilon-\epsilon_p)}{\xi} > 3$, but there is a definite disagreement over the rest of the curves; Williams's curve falls off more steeply on the negative side of the peak and less rapidly on the positive side. It is to be presumed that the Landau curve is correct since the method used to obtain it is rigorous, and since it agrees with the calculations to be presented later. Landau's value for the constant j is 0.37, which represents a hardly discernible shift in the position of the peak as compared with Williams's value (j=0.30). The values of the constants A and ϵ, according to the results of the relativistic quantum theory, are (3),(10):

$$A = \frac{2\pi N e^4}{\mu_e \beta^2} \frac{Z}{A} \rho \qquad 2.5$$

$$\epsilon = \frac{(1-\beta^2) I_Z^2}{2\mu_e \beta^2} e^{\beta^2} \qquad 2.6$$

where

N = Avogadro's number = 6.024×10^{23} mole^{-1}

μ_e = rest energy of electron = 5.109×10^5 ev.

I_Z = mean ionization potential of atomic electrons $\doteq 13.5 Z$ ev.

$\frac{Z}{A}$ = mean ratio of atomic number to atomic weight for material traversed.

ρ = density of material in gm/cm^3

$\beta^2 = v^2/c^2$, where v is the velocity of the incident particle (assumed to have a charge $\pm e$.)

The first complete quantum mechanical treatment of the collision of fast electrons with atoms was given in 1930 by Bethe (12); this treatment neglected relativistic corrections. A method of treating the relativistic interaction of two electrons in the quantum theory was developed in 1932 by Møller (11), and applied to collisions of fast electrons with free and bound electrons. A detailed study of the collisions of electrons with atoms along the lines of (12), but using Møller's methods to take account of retardation and relativistic effects is contained in a second paper of Bethe (13); the general method is summarized in Bethe's article in the *Handbuch der Physik* (18). Møller's methods were applied by Bhabha to the calculation of the collision cross sections for positrons with electrons (17), and for mesons of spin 0 and spin $\frac{1}{2}$ (14). Collision cross sections for mesons of spin $\frac{1}{2}$ and spin 1 with electrons have been calculated by Massey and Corben (15), and for spin 1 also by Oppenheimer, Snyder, and Serber (16). All these results are summarized in a form convenient for the present purpose in a review article by Rossi and Greisen (4,p.243-7); range-momentum curves for mesons or protons with $0.5 < p/\mu < 10$ are given in their paper. A thorough discussion of the range-energy relation at low and moderate energies may be found in a review article by Livingston and Bethe (10,Sec.95). Curves for range vs energy for protons and α-particles, and curves giving the mean rate of energy loss per cm, at energies up to 15 Mev, are presented, based on an analysis of the experimental and theoretical results up to that time (1937).

In review, we may say that satisfactory answers are available to the first problem mentioned in Section 1, that of determining the function $\chi(E')$, and that a satisfactory solution, that of Landau (3), is available for the second problem as applied to high energy mesons for small thicknesses of material. The determination of $P(E,t)$ when terms of the order of ξ/E'_m can no longer be neglected, and when account is taken of the variations in $\chi(E')$ with the decreasing kinetic energy of the meson, is to be attempted in the present thesis.

3. Outline of Thesis.

In Chapter II, a sketch is given of the methods by which cross sections for meson electron collisions are computed using relativistic quantum theory. The results for collisions of electrons, positrons, and mesons of spin 0, spin $\frac{1}{2}$, and spin 1 are summarized in the form in which they will be needed later. It is pointed out that differences due to spin are unimportant except for meson energies of the order of 20 Bev or greater, where it is not certain that the present theory is correct (4,p.245). The formulas for the scalar meson will therefore be used in the later calculations, and the results will be applicable to mesons of any spin within the range of energies treated in this thesis (1 Mev to 1000 Mev for mesons of 200 electron masses). Chapter II closes with a discussion of the importance of other modes of energy loss (collision with nuclei, radiation) and their effect on the accuracy of the results.

A discussion of the system of units to be used in the remainder of the thesis will be found at the beginning of Chapter III. The system of units, and definitions of all special symbols used throughout are collected for reference in Tables 1 and 2 at the end of the thesis. In Section 9, the fundamental integro-differential equation governing the probability function $P(\mathcal{E},t)$, (which will be called the "energy loss distribution"), is derived. The treatment of this equation falls into two parts, corresponding to Chapters III and IV. Chapter III is concerned with the case when the function $\chi(E')$, (which will be called the "collision spectrum"), is independent of the energy E_k of the incident particle (over the range of energies E_k^0 to $E_k^0-\mathcal{E}$, where $\mathcal{E}$ is the energy lost). The problem can then be solved rigorously. The general solution for $P(\mathcal{E},t)$ is derived in terms of a Laplace transform, for the case where $\chi(E')$ is independent of E_k but may depend on the depth t in the material. The remainder of Chapter III is concerned with the inversion of this Laplace transform. The special case when Landau's assumptions are valid (t small, see Section 2) is discussed, and the formula for the Laplace transform of ϕ is derived. The opposite case when t is large, and the distribution in energy loss is nearly Gaussian, can be treated by an expansion in terms of the Gauss function and its derivatives (Edgeworth, or Gram-Charlier expansion); this is essentially an asymptotic expansion in powers of $t^{-1/2}$. The general relations are derived between the moments of the collision spectrum $\chi(E')$ and the semi-invariants and moments of the energy loss distribution $P(\mathcal{E},t)$. These relations were already known and have been discussed by Cramèr (19). They are valid for all values of t, but the Edgeworth expansion based on them is only useful for large values of t.

A modification of the Edgeworth expansion is presented, based on the connection between the Edgeworth expansion and the saddle-point method, which gives a series asymptotically valid for large values of t, but also useful when t is small. By this means it is possible to locate the position $\mathcal{E}_p$ of the peak of the curve $P(\mathcal{E},t)$ very accurately, even in the limiting case of very small t (Landau curve), and to get a fairly good approximation to the shape of the curve. The modified Edgeworth expansion leads naturally to the introduction of a set of weighted moments computed after multiplying $P(\mathcal{E},t)$ by an exponential weight factor which emphasizes values of $\mathcal{E}$ near the peak $\mathcal{E}_p$ and reduces the tail of the curve for larger values of $\mathcal{E}$. These weighted moments are better suited to characterize the curves for small values of t than the ordinary unweighted moments, and reduce to the unweighted moments for large values of t. Other methods are outlined, similar to those used by Williams (2), which are useful in determining $P(\mathcal{E},t)$ when $\chi(E')$ differs slightly from a function $\chi^0(E')$ for which the energy loss distribution is known.

All these methods, together with the results of Landau, are finally combined in the calculation of a family of distribution curves $P(\mathcal{E},t)$ and $\int_{\mathcal{E}}^{\infty} P(\mathcal{E},t)\,d\mathcal{E}$. (See Fig.2.) These curves are characterized by a single parameter, the "weighted skewness", and they extend from the limiting case of small

(Landau curve) to that of large t (Gaussian curve). It is shown that, to a sufficient degree of approximation, any $P(\epsilon,t)$ can be represented by one of the curves of this family if the weighted skewness, the most probable energy loss ϵ_p, and the appropriate energy loss scale factor (the weighted fluctuation), are properly chosen. This result holds whenever $P(\epsilon,t)$ is a solution corresponding to a function $\chi(E')$ of the form derived for incident scalar mesons, and when the dependence of $\chi(E')$ on the energy ϵ lost by the meson is neglected. It holds in spite of the fact that the form of $\chi(E')$ is somewhat different for different initial meson energies, although this difference must of course be taken into account in choosing the appropriate skewness, ϵ_p, and scale factor. Curves are given for the determination of these three "weighted" parameters for any initial energy and thickness, (within the ranges for which the approximations made are valid), and for determining the weighted parameters from the unweighted parameters ($\bar{\epsilon}$, $(\overline{(\Delta\epsilon)^2})^{1/2}$, $\overline{(\Delta\epsilon)^3}/(\overline{(\Delta\epsilon)^2})^{3/2}$), and vice versa.

If we make the plausible assumption that when the variation of $\chi(E')$ with ϵ is taken into account, the curves of Chapter III may still be used provided the parameters are properly chosen, it remains only to determine the parameters in order to have a solution for all thicknesses t. In Chapter IV, equations of motion for the unweighted moments $\bar{\epsilon}$, $\overline{(\Delta\epsilon)^2}$, $\overline{(\Delta\epsilon)^3}$, etc., are derived, which are valid when $\chi(E')$ is a function also of ϵ. These equations can be solved explicitly if terms involving higher than first derivatives of χ with respect to ϵ are neglected. The resulting solutions are valid as long as $\left(\overline{(\Delta\varepsilon)^2}\right)^{1/2} << \mathcal{E}_k$, i. e. up to about 90% of the range for mesons. (Formulas previously given for $\overline{(\Delta\varepsilon)^2}$ by Bohr (1) and quoted by Bethe and Livingston (10) neglect also the first derivatives of $\varkappa$, and are valid only as long as $\bar{\varepsilon} << \mathcal{E}_k$.) Curves are given from which the three unweighted parameters may be determined for meson energies from 1 Mev to 1000 Mev. These may then be used with the curves of Chapter III to determine $P(\varepsilon,t)$. The curves are also applicable to protons from about 10 Mev to 10 Bev.

Chapter V begins with a discussion of distributions related to $P(\varepsilon,t)$, such as the distribution in distances traversed in losing a given amount of energy, in particular, the distribution in range of particles of given initial energy; and the probability distribution of the initial energy, given the depth and final energy, or given the range. It is shown that all these distributions can be derived if $P(\varepsilon,t)$ is known. Formulas for the range, fluctuation in range, and skewness in range distribution are derived, and curves are given. The formulas for the first two of these quantities were previously derived by Bohr (1), and are quoted by Bethe and Livingston (10). Chapter V closes with a discussion of extensions and other application of the methods used, and a summary of the results obtained. A set of instructions for the use of the curves is given, and a specific example is worked out.

CHAPTER II

COLLISION CROSS SECTIONS

4. Meson and Free Electron.

The first satisfactory relativistic treatment of the interaction of two charged particles is due to Møller (11), who calculated the collision cross section for two electrons, to a first approximation in an expansion in powers of the electronic charge e, the first non-vanishing term in the interaction being of order e^4. The method is described by Heitler (20,Sec.10), who carries it to the point of obtaining the complete matrix element of the interaction energy for the collision of two electrons. The cross section is then readily obtained from the usual perturbation theory formula for the probability of the corresponding transition.

The electromagnetic field of two charged particles can be separated into a longitudinal and a transverse part (20). The longitudinal part gives rise to the ordinary electrostatic interaction whose matrix element is readily calculated. The interaction due to the transverse part of the field must be treated as a two step process in which one particle first emits a light quantum, which is then absorbed by the other particle. The matrix element for this two step process is calculated using the second order perturbation theory, and added to the matrix element of the electrostatic interaction energy to obtain the total matrix element for the collision process.

The method is readily applied to the interaction of any two charged particles for each of which the expression for the interaction with the electromagnetic field is known. The result, according to Bhabha (14), for the cross section for a collision of a meson of spin 0 with a stationary free electron, in which the meson loses an amount of energy between E' and $E'+dE'$ is:

$$d\sigma = \pi r_0^2 \frac{2\mu_e}{\beta^2}\frac{dE'}{E'^2}\left[1-\beta^2\frac{E'}{E'_m}\left(1-\frac{1}{2}\left(\frac{2\mu_e}{\mu}\right)^2\frac{1}{1+\frac{2\mu_e}{\mu^2}E}\right)\right] \qquad 4.1$$

where:

$r_0 = e^2/mc^2 =$ classical radius of the electron

$\mu_e = mc^2 =$ rest energy of electron

$\mu =$ rest energy of meson

$\beta = v/c$ for meson before collision

$E =$ total energy of meson before collision

$E' =$ energy lost by meson in collision

$E'_m = 2\mu_e \dfrac{E^2/\mu^2 - 1}{1+\frac{2\mu_e}{\mu^2}E+\frac{\mu_e^2}{\mu^2}} =$ maximum value of E' allowed by conservation laws

Since $\frac{\mu_e}{\mu} \sim 0.005$ for mesons, the terms in $(\mu_e/\mu)^2$ in 4.1 are negligible, and we may write:

$$d\sigma = \pi r_0^2 \frac{2\mu_e}{\beta^2} \frac{dE'}{E'^2}\left[1-\beta^2 \frac{E'}{E'_m}\right] \qquad 4.2$$

$$E'_m = 2\mu_e \frac{E^2/\mu^2 - 1}{1 + \frac{2\mu_e}{\mu^2} E} \qquad 4.3$$

The corresponding formula for mesons of spin $\frac{1}{2}$ (or protons) was also given by Bhabha (14), and by Massey and Corben (15), and is, according to Rossi and Greisen (4):

$$d\sigma = \pi r_0^2 \frac{2\mu_e}{\beta^2} \frac{dE'}{E'^2}\left[1-\beta^2 \frac{E'}{E'_m} + \frac{1}{2}\left(\frac{E'}{E+\mu}\right)^2\right] \qquad 4.4$$

The formula for mesons of spin 1 was given by Massey and Corben (15) and by Oppenheimer, Snyder, and Serber, (16), and is, according to (4):

$$d\sigma = \pi r_0^2 \frac{2\mu_e}{\beta^2} \frac{dE'}{E'^2}\left[\left(1-\beta^2 \frac{E'}{E'_m}\right)\left(1+\frac{1}{3}\frac{E'}{E_c}\right) + \frac{1}{3}\left(\frac{E'}{E+\mu}\right)^2\left(1+\frac{1}{2}\frac{E'}{E_c}\right)\right] \qquad 4.5$$

where:

$E_c = \mu^2/\mu_e \doteq$ 2 x 10^{10} ev, for mesons.

It will be observed that 4.3, 4.4, and 4.5 differ only for large values of E'. For the limit of very small and very large energies, the maximum energy loss E'_m becomes:

$$E'_m \doteq \frac{4\mu_e}{\mu} E_k \qquad (E_k << \mu)$$

$$\doteq 0.02\, E_k \qquad \text{(mesons)} \qquad 4.6$$

$$E'_m \doteq E \qquad (E >> E_c) \qquad 4.7$$

Thus only for very high energies can E' be large enough to result in a significant difference between the cross sections for different spins.

5. Meson and Atom.

The collision of a fast charged particle with an atom can be treated approximately using the Born approximation. This has been done, using non-relativistic quantum theory, by Bethe (12), (18), who finds, for the cross section for a collision of a fast charged particle with an atom, in which the atom is left in a particular excited state n and the incident particle loses an amount of momentum between $\hbar K$ and $\hbar(K+dK)$, the formula:

$$\sigma_n(K)\,dK = \frac{8\pi\mu^2 e^4}{c^4\hbar^2}\,\frac{dK}{K^3 p^{\circ 2}}\left|\epsilon_{on}(K)\right|^2 \qquad 5.1$$

where:

$$\epsilon_{on}(K) = \sum_{s=1}^{Z} \int e^{i\vec{K}\cdot\vec{r}_s}\, \psi_o(\ldots\vec{r}_s\ldots)\, \psi_n^*(\ldots\vec{r}_s\ldots)\ldots d\tau_s\ldots$$

and:

μ = rest energy of incident particle

p° = initial momentum of incident particle

$\vec{r}_s$ = position coordinate of sth atomic electron

$\psi_0(\ldots\vec{r}_s\ldots) =$ wave function of atom in initial (ground) state

$\psi_n(\ldots\vec{r}_s\ldots) =$ wave function of atom in final state

Evidently if K is large compared with the momenta of the atomic electrons, then ϵ_{0n} will be very small unless one of the electrons in the final state has a momentum differing from $\hbar\vec{K}$ by an amount not much greater than the initial momentum of that electron, i.e., unless one of the atomic electrons acquires practically all the momentum given up by the incident particle. The cross section is then essentially the same as if this electron alone were present, and the total momentum of the incident particle and this one electron is conserved. Thus we may expect, as seems reasonable on physical grounds, that collisions in which an atomic electron receives an energy large compared with its binding energy may be treated as if the electron were free. For such collisions, the formulas of Section 4 are valid. The energy lost by the incident particle is of course in all cases equal to the excitation energy of the final state of the atom.

Let us set:

$$Q = \frac{c^2\hbar^2K^2}{2\mu_e}$$

$$\frac{dK}{K^3} = \frac{c^2\hbar^2}{4\mu_e}\frac{dQ}{Q^2} \qquad 5.2$$

Then, integrating 5.1 over Q, we have for the total cross section for excitation of the nth atomic state:

$$\sigma_n = \pi r_0^2 \frac{2\mu_e}{\beta^2}\left(\frac{\mu^2\beta^2}{c^2 p_0^2}\right)\int \frac{dQ}{Q^2}\left|\epsilon_{0n}(Q)\right|^2 \qquad 5.3$$

($\beta^2 = V^2/c^2$ for the incident particle.) The dependence of σ_n on the excitation energy E' is complicated and varies from one kind of atom to another, except when E' is large compared with the binding energies of the atomic electrons when the formulas of Section 4 are valid.

The average energy loss per cm in a material of density ρ, atomic weight A, due to collisions with atoms will be:

$$-\frac{dE}{dt} = N\frac{\rho}{A}\sum_n E_n \sigma_n \qquad 5.4$$

(N = Avogadro's number = 6.02 x 10^{23} atoms/mole.)
For hydrogen, Bethe (12) has carried out this summation numerically, and finds, for the total rate of energy loss by ionization and excitation:

$$-\frac{dE}{dt} = 2\pi r_0^2 N \frac{\rho}{A}\frac{2\mu_e}{\beta^2}\log\left(\frac{2\mu_e\beta^2}{1.105 Rh}\right) \qquad 5.5$$

For complex atoms, 5.4 can be evaluated by inverting the order of integration over Q and summation over n, and using a sum rule derived by Bethe (12) which is a generalization of the sum rule for the matrix elements ϵ_{0n} which occur in the theory of optical transitions. The result is that if we define a suitable average ionization potential I_Z, we can write for any kind of atom:

$$-\frac{dE}{dt} = 2\pi r_0^2 N \frac{Z}{A}\rho\frac{2\mu_e}{\beta^2}\log\frac{2\mu_e\beta^2}{I_Z} \qquad 5.6$$

Bloch (37) has arrived at a similar formula for collisions of a charged particle with a Fermi-Thomas statistical atom.

In Bloch's formula, I_z is replaced by $\kappa R h Z$, where κ is a number whose value has not yet been calculated. But Bloch showed that the Z dependence of I_z as given by his formula agrees with the experimental results for the stopping of fast electrons. According to Bloch, I_z in 5.6 is given to a sufficient approximation by:

$$I_z = 13.5\, Z \text{ ev} \tag{5.7}$$

The correct relativistic generalization of 5.6 is (13), (18):

$$-\frac{dE}{dt} = 2\pi r_0^2 N \frac{Z}{A}\rho \frac{2\mu_0}{\beta^2}\left[\log\frac{2\mu_0\beta^2}{(1-\beta^2)I_z} - \beta^2\right] \tag{5.8}$$

5.8 is valid for an incident particle of any spin provided its energy is not too high.

Bethe (12) has shown that the quantity:

$$\sum_n E_n^2 \sigma_n \tag{5.9}$$

can also be evaluated by inverting the order of summation over n and integration over Q, and using the generalized sum rule. The method is outlined, and the result given, in a review article by Livingston and Bethe (10, p. 282). This quantity is important in determining the fluctuations in energy loss, as will be shown later. Except at low energies of the incident particle, the principal contributions to the sum 5.9 will come from collisions for which E' is large compared with the ionization potentials, so that 4.2 can be used to evaluate it.

6. The Collision Spectrum.

The most convenient way of expressing the results of the two preceding sections, for purposes of the developments to follow, is in terms of the "collision spectrum" $\chi(E',E)$. The collision spectrum is a function characteristic of the ionizing particle and the material through which it is passing.* It is defined by requiring that:

$\chi(E',E)\,dE'\,dt$ = probability that in going a distance dt a particle of total energy E will suffer a collision in which it loses an amount of energy between E' and $E'+dE'$. 6.1

The number of atoms per square centimeter in the path of a particle traversing a thickness dt of material of density ρ and atomic weight A is:

$$\frac{N}{A}\rho\,dt$$

where N is Avogadro's number. If the deBroglie wave length of the particle is short compared with the interatomic distances, then the probability 6.1 is just the sum of the probabilities for collisions with the individual atoms:

$$\chi(E',E)\,dE'\,dt = \sigma(E',E)\,dE'\,\frac{N}{A}\rho\,dt \qquad 6.2$$

Using the formula 4.2, we have, for E' large compared with the ionization potentials of all electrons in the atoms, and for mesons of spin 0:

$$\chi(E',E) = \pi r_0^2 N \frac{Z}{A}\rho\,\frac{2\mu_e}{\beta^2}\,\frac{dE'}{E'^2}\left[1-\beta^2\frac{E'}{E'_m}\right] \qquad 6.3$$

---

* It may also be a function of position in the material if the material is inhomogeneous.

(For $E' < E'_m$. $\chi(E',E) = 0$ for $E' > E'_m$.) For mesons (or protons) of spin ½, we have, by 4.4:

$$\chi(E',E) = \pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2} \frac{dE'}{E'^2} \left[1 - \beta^2 \frac{E'}{E'_m} + \frac{1}{2}\left(\frac{E'}{E+\mu}\right)^2\right] \qquad 6.4$$

For mesons of spin 1, by 4.5:

$$\chi(E',E) = \pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2} \frac{dE'}{E'^2} \left[\left(1 - \beta^2 \frac{E'}{E'_m}\right)\left(1 + \frac{1}{3}\frac{E'}{E_c}\right) + \frac{1}{3}\left(\frac{E'}{E+\mu}\right)^2 \left(1 + \frac{1}{2}\frac{E'}{E_c}\right)\right] \qquad \left(E_c = \frac{\mu^2}{\mu_e}\right) \qquad 6.5$$

For collisions in which E' is comparable with the ionization potentials, we have no simple formula for $\chi(E',E)$, but according to 5.8, we have:

$$\int_0^\infty E' \chi(E',E)\, dE' = 2\pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2}\left[\log \frac{2\mu_e \beta^2}{(1-\beta^2) I_z} - \beta^2\right] \qquad 6.6$$

If η is an energy large compared with the ionization potentials of the atomic electrons, we have, by 6.3 (for spin 0, or any spin if E is not too large):

$$\int_\eta^\infty E' \chi(E',E)\, dE' = \pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2}\left[\log \frac{E'_m}{\eta} - \beta^2\right] \qquad 6.7$$

When $\frac{2\mu_e}{\mu^2} E \ll 1$, we have, by 4.3:

$$E'_m \doteq 2\mu_e \left(\frac{E^2}{\mu^2} - 1\right) = 2\mu_e \frac{\beta^2}{1-\beta^2} \qquad 6.8$$

In this case:

$$\int_\eta^\infty E' \chi(E',E)\, dE' = \pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2}\left[\log \frac{2\mu_e \beta^2}{(1-\beta^2)\eta} - \beta^2\right]$$

Comparing this with 6.6, we have:

$$\int_0^\eta E' \chi(E',E)\, dE' = \pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2}\left[\log \frac{2\mu_e \beta^2 \eta}{(1-\beta^2) I_z^2} - \beta^2\right] \qquad 6.9$$

Formula 6.9 depends only on the velocity of the incident particle, and not on its mass. We may expect that the probability of small energy transfers E' for any fast particle will depend only on its velocity (and charge), and not on its mass or spin. Since for any value of β^2, and any spin, we can choose μ large enough so that the assumptions made in deriving 6.9 are valid, we may suppose that 6.9 is valid for mesons of any spin or velocity, provided η is not too large. (η must, however, be large compared with the ionization potentials of the atomic electrons.)

Formula 6.2 neglects interference effects between the de Broglie waves scattered from different atoms. The de Broglie wavelength for a meson at low velocities is:

$$\lambda\!\!\!^{-} = \frac{\hbar}{p} \doteq \frac{\hbar c}{\sqrt{2\mu E_k}} = \frac{\hbar}{mc}\frac{1}{\sqrt{2}}\frac{\mu_e}{\mu}\sqrt{\frac{\mu}{E_k}} \qquad 6.10$$

$$= 0.00263\frac{\mu_e}{\mu}\sqrt{\frac{\mu}{E_k}}\ \text{Å}$$

The spacing between atoms in a solid is of the order of 0.5 Å, so that the condition that $\lambda\!\!\!^{-}$ be small compared with the atomic spacing is well satisfied for mesons ($\frac{\mu_e}{\mu} \sim 0.005$) down to energies $E_k \sim 10^{-5}\mu$, i.e. down to meson energies of only 1000 ev.

Formulas 6.3 - 6.9 must be modified when E'_m is comparable with the ionization potentials of the innermost atomic electrons. This case is discussed in detail by Bethe and Livingston (10, Chap.XVI). For the present purpose it will be a good enough approximation to replace the factor Z in these formulas by the "effective number of electrons" Z'. The latter may be defined as the total oscillator strength of all the electrons for which the ionization potential is less than $\frac{1}{4}E'_m$. The ionization potentials of the inner electrons may be taken to be:

$$I_{n,z} \doteq 13.5 \frac{Z^2}{n^2} \text{ ev} \qquad 6.11$$

where there are 2 electrons with $n = 1$, 8 with $n = 2$, etc. The innermost electrons have an average oscillator strength of about 0.6 apiece. At low energies:

$$E'_m \doteq 4 \frac{\mu_e}{\mu} E_k \qquad 6.12$$

For 1 Mev mesons (E'_m = 40,000 ev) traversing lead, only the inner two shells (K,L) have a binding energy greater than $\frac{1}{4}E'_m$. The oscillator strength of these 10 electrons is 6, so we should use Z' = 76, instead of 82, a correction of 7%. For higher energies or materials of lower atomic number, the correction is even less. This correction to the effective Z will be ignored in the remainder of this thesis. If necessary, it can easily be applied.

Formulas 6.4 and 6.5 differ only negligibly from 6.3 except when the energy loss E' in a single collision is comparable with the total energy E. This can only occur for large energies E of the meson, and within the range of energies to be studied in this thesis, 6.3 is a sufficiently good approximation for mesons of any spin. To verify this, we compute the correction to the third moment of the collision spectrum, which is the highest moment that is important in later computations. For mesons of spin $\frac{1}{2}$, the third moment is:

$$\int_0^{E'_m} E'^3 \chi(E',E)\,dE' = \pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2} E'^2_m \left[\frac{1}{2} - \frac{1}{3}\beta^2 + \frac{1}{8}\left(\frac{E'_m}{E+\mu}\right)^2\right] \qquad 6.13$$

The last term in the brackets is the correction due to spin. For moderate energies, we can use equation 6.8; for larger energies, E'_m is even smaller. Substituting 6.8 in 6.13:

$$\int_0^{E'_m} E'^3 \chi(E',E)\,dE' = \pi r_0^2 N \frac{Z}{A} \rho \frac{2\mu_e}{\beta^2} E'^2_m \left[\frac{1}{2} - \frac{1}{3}\beta^2 + \frac{1}{8}\left(\frac{2\mu_e}{\mu}\right)^2 \left(\frac{E_k}{\mu}\right)^2\right] \qquad 6.14$$

At $\frac{E_k}{\mu} = 10$, the correction due to spin is only 0.75%, for mesons ($\mu \doteq 200\mu_e$). For mesons of spin 1, we have:

$$\frac{E'_m}{E_c} \doteq \frac{\frac{2\mu_e}{\mu^2}(E^2-\mu^2)}{\mu^2/\mu_e} = \frac{1}{2}\left(\frac{2\mu_e}{\mu}\right)^2\left(\frac{E^2}{\mu^2}-1\right)$$

$$= 5 \times 10^{-5}\left(\frac{E^2}{\mu^2}-1\right) \qquad 6.15$$

$$= 0.006 \quad \left(\text{for } \frac{E}{\mu} = 11\right)$$

Therefore the terms in E'/E_c in 6.5 can be neglected for $E_k \leq 10\mu$. The corrections for spin 1 then become practically the same as for spin $\frac{1}{2}$, and can therefore be neglected within the range of energies to be studied. The spin corrections are open to some doubt anyway, as they involve the assumption of the usual equations for the electromagnetic field of the meson down to distances smaller than 10^{-13} cm from its center (4,p.245.)

7. Other Modes of Energy Loss.

Formulas for the probability of energy loss by radiation, as computed for mesons by Christy and Kusaka (36), are summarized by Rossi and Greisen (4,p.256). A table of the total energy loss by radiation, in comparison with the loss by collisions, for mesons, protons, and electrons of various energies and in various materials, has been given by Bhabha (14). For mesons, the total energy loss by radiation is negligible except at very high energies, amounting to about one percent of the loss by collisions at 1000 Mev in lead, and at 10,000 Mev in water. For protons, the energy loss by radiation is even smaller, being entirely negligible up to 10,000 Mev in lead, and up to 10^6 Mev in water. Therefore the effect of radiation on the average energy loss of mesons or protons can be neglected within the energy range to be studied here. (Up to 1000 Mev for mesons, or 10,000 Mev for protons.)

In estimating the effect of radiation losses on the fluctuations in energy loss, we consider first those radiation processes in which the radiated photon has an energy less than E_m'. Designating the energy lost in a single radiation process by E'', we define the fractional energy loss v by:

$$v = \frac{E''}{E} \qquad 7.1$$

Let $\phi(v)dv$ be the probability per unit distance of a fractional energy loss between v and $v+dv$. For the present purpose, it will be a sufficiently good approximation to set:

$$\phi(v) = \frac{K(E)}{vE} \tag{7.2}$$

The total average energy loss per unit distance by radiation is then:

$$\int_0^1 vE\,\phi(v)\,dv = K(E) \tag{7.3}$$

The function $K(E)$ can be determined from the table given by Bhabha (14). The contribution of radiation processes for which $E'' \leqq E'_m$ to the third moment of the energy loss spectrum is:

$$\int_0^{E'_m/E} (vE)^3 \phi(v)\,dv = \frac{1}{3} E'^3_m \frac{K(E)}{E} \tag{7.4}$$

Using the approximation 6.8, this becomes:

$$\int_0^{E'_m/E} (vE)^3 \phi(v)\,dv \doteq \frac{1}{3} (2\mu_e)^3 \left(\frac{E^2}{\mu^2}-1\right)^3 \frac{K(E)}{E} \tag{7.5}$$

For 1000 Mev mesons ($\frac{E}{\mu}=11$), $K(E)$ in lead, according to Bhabha (14), is 0.17 Mev/cm, so that 7.5 becomes:

$$\int_0^{E'_m/E} (vE)^3 \phi(v)\,dv = 100\,(2\mu_e)^3\ \text{cm}^{-1} \quad (E_k = 1000\ \text{Mev in lead}) \tag{7.6}$$

At the same energy, in lead, 6.3 yields:

$$\int_0^{E'_m} E'^3 \chi(E',E)\,dE' \doteq \pi r_0^2 N \frac{Z}{A} \rho\,(2\mu_e)^3 \left(\frac{E^2}{\mu^2}-1\right)^2 \frac{3-2\beta^2}{6\beta^2}$$

$$= 4800\,(2\mu_e)^3\ \text{cm}^{-1} \quad (E_k = 1000\ \text{Mev in lead}) \tag{7.7}$$

The correction to 7.7 due to 7.6 amounts to only 2%. For lower energies or material of lower atomic number, the correction would be even less. Therefore the effect of radiation losses for which $E'' \leq E'_m$ can be neglected within the range of meson energies to be studied.

The number of radiation processes per unit distance for which $E'' > E'_m$, is:

$$\int_{E'_m/E}^{1} \phi(v)\,dv = \frac{K(E)}{E} \log E/E'_m$$

$$\doteq \frac{K(E)}{E} \log \frac{E}{2\mu_e} \frac{\mu^2}{(E^2-\mu^2)}$$

$$= 4 \times 10^{-4}\ \text{cm}^{-1}\ (E_k = 1000\ \text{Mev in lead}) \qquad 7.8$$

The range of 1000 Mev mesons in lead is only about 66 cm, so that less than 3% of the mesons will suffer any such radiation loss at all. Those particles which suffer a loss $E'' > E'_m$ in going a distance t will in general suffer a total energy loss $\mathcal{E}$, due to radiation and collisions, which is greater than any particle is likely to suffer due to collision losses alone. Hence if the distribution $P(\mathcal{E},t)$ in energy loss $\mathcal{E}$ due to collisions alone is determined, the effect of radiation losses will be to reduce the function $P(\mathcal{E},t)$ by less than 3% for all values of $\mathcal{E}$ for which $P(\mathcal{E},t)$ has an appreciable value, and to add a very small probability to $P(\mathcal{E},t)$ for large values of $\mathcal{E}$. In other words, more than 97% of the mesons will be distributed according to the uncorrected function $P(\mathcal{E},t)$, the

The correction to 7.7 due to 7.6 amounts to only 2%. For lower energies or material of lower atomic number, the correction would be even less. Therefore the effect of radiation losses for which $E'' \leq E'_m$ can be neglected within the range of meson energies to be studied.

The number of radiation processes per unit distance for which $E'' > E'_m$, is:

$$\int_{E'_m/E}^{1} \phi(v)\,dv = \frac{K(E)}{E}\log \frac{E}{E'_m}$$

$$\doteq \frac{K(E)}{E}\log \frac{E}{2\mu_e}\frac{\mu^2}{(E^2-\mu^2)}$$

$$= 4 \times 10^{-4}\ \text{cm}^{-1} \quad (E_k = 1000\ \text{Mev in lead}) \qquad 7.8$$

The range of 1000 Mev mesons in lead is only about 66 cm, so that less than 3% of the mesons will suffer any such radiation loss at all. Those particles which suffer a loss $E'' > E'_m$ in going a distance t will in general suffer a total energy loss $\mathcal{E}$, due to radiation and collisions, which is greater than any particle is likely to suffer due to collision losses alone. Hence if the distribution $P(\mathcal{E},t)$ in energy loss $\mathcal{E}$ due to collisions alone is determined, the effect of radiation losses will be to reduce the function $P(\mathcal{E},t)$ by less than 3% for all values of $\mathcal{E}$ for which $P(\mathcal{E},t)$ has an appreciable value, and to add a very small probability to $P(\mathcal{E},t)$ for large values of $\mathcal{E}$. In other words, more than 97% of the mesons will be distributed according to the uncorrected function $P(\mathcal{E},t)$, the remaining few having suffered abnormally large energy losses due to radiation. Such energy losses are also likely to be associated with abnormally large angular deflections.

The correction to the second and third moments of the energy loss spectrum due to radiation losses for which $E'' > E'_m$ may be considerable, but this correction should not be made if these moments are to be used as in Chapters III and IV to determine the shape of the main portion of the distribution curve $P(E,t)$. The curve $P(E,t)$ should first be determined as in Chapters III and IV, neglecting radiation losses entirely, or, at very high energies, taking radiation losses $E'' \leq E'_m$ into account. A correction for the remaining radiation losses can then be made by assuming a probability for a single additional loss in energy given by 7.2, or by the more accurate formulas given by Rossi and Greisen (4). This procedure will give a good approximation to $P(E,t)$ so long as the average energy loss by radiation is small compared with the loss by collision.

A meson may also lose energy by its interaction with atomic nuclei. Very little is definitely known about the cross section for this kind of energy loss, either theoretically or experimentally. Apparently no experimental evidence for this kind of energy loss has yet been found (except for capture of the meson by the nucleus). At the lowest energies to be studied in this thesis (1Mev), the meson wave length is only about 10^{-13} cm, so that the cross section for any interaction with the nucleus can hardly be greater than the geometrical cross section of the nucleus, i.e. about 10^{-24} cm^2. For a cross section of 10^{-24} cm^2, the number of processes taking place per unit distance is:

$$\frac{N}{A}\rho \times 10^{-24}\,cm^2 \sim 0.03\ cm^{-1}\ (\text{lead or water}) \qquad 7.9$$

The number of such processes is thus less than about 0.03 cm^{-1} for almost any solid or liquid. The fact that such processes have not been observed suggests that the number is actually considerably smaller. In the absence of further information, it is impossible to determine the effect of nuclear interactions on the energy loss. We may expect that collisions in which a meson loses an appreciable amount of energy due to its interaction with the nuclear particles will be comparatively rare, and probably associated with abnormally large angular deflections.

CHAPTER III

THE FUNDAMENTAL EQUATION AND ITS SOLUTION FOR A CONSTANT COLLISION SPECTRUM

8. Units and Notation.

From now on, we will use the following system of units, convenient for the discussion of particles of cosmic ray energies:

Table 1

System of Units

Quantity	Unit	Symbol
Fundamental Units:		
Length	Centimeter	cm
Velocity	velocity of light	c
Energy	electron-volt	ev
Derived or Special Units:		
Mass	electron-volt/c^2	ev/c^2
Density of braking material	gram per cubic centimeter	gm/cm^3
Path length in braking material	collision unit	$[\pi r_0^2 N \frac{Z}{A}\rho]^{-1}$ $=[0.150 \frac{Z}{A}\rho]^{-1}$

We will use the following symbols, most of which have been introduced earlier:

Table 2

Partial List of Symbols

Quantity	Symbol	Value
Mass of electron	μ_e	5.109×10^5 ev/c^2
Mass of ionizing particle	μ	$\sim 200\mu_e$ (meson)

Quantity	Symbol	Value
Mass ratio factor	γ	$200\,\mu_e/\mu$ (1 for meson)
Avogadro's number	N	6.024×10^{23} mole^{-1}
Classical radius of electron	r_0	$e^2/\mu_e = 2.818 \times 10^{-13}$ cm
Density of braking material in gm/cm^3	ρ	
Atomic weight of braking material in gm mole^{-1}	A	
Atomic number of braking material	Z	
Path length of ionizing particle in cm	t	
Path length of ionizing particle in collision units	x	$\pi r_0^2 N \frac{Z}{A} \rho t = 0.150 \frac{Z}{A} \rho t$
Kinetic energy of ionizing particle	E_k	
Total energy of ionizing particle	E	$E_k + \mu$
Energy lost in a single collision	E'	
Maximum energy which can be lost in a single collision	E'_m	$2\mu_e\left(\frac{E^2}{\mu^2}-1\right)/\left(1+0.01\gamma\frac{E}{\mu}\right)$
Total energy loss	ε	$\sum E'$
Velocity of ionizing particle	β	$(1-\mu^2/E^2)^{1/2}$

Additions will be made to this table from time to time. The complete list will be found at the end of the thesis.

The collision unit has been introduced as a measure of the path length in order to eliminate numerical factors from the formulas. For materials with more than one kind of atom, Z/A is to be taken as the average value or as

$\Sigma Z/\Sigma A$ for a molecule. The path length x in collision units is a dimensionless quantity and represents the average number of electrons, thought of as spheres of radius r_0, through which a random straight line of length t will pass. For a given path length x, the mean energy lost by a meson is nearly independent of the kind of material traversed. With this notation, the probability that in going a distance dx a particle suffers a collision in which it loses an amount of energy between E' and $E'+dE'$, may be written (Sec.6):

$$\chi(E')\,dE'dx = \begin{cases} 0 & E' > E'_m \\ \dfrac{2\mu_e\,dx}{\beta^2}\dfrac{dE'}{E'^2}\left(1-\beta^2\dfrac{E'}{E'_m}\right) & \eta \le E' \le E'_m \end{cases} \qquad 8.1$$

$$\left[\int_0^\eta E'\chi(E')\,dE'\right]dx = \frac{2\mu_e\,dx}{\beta^2}\log \eta/\epsilon \qquad E'<\eta$$

where ϵ is given by 2.6, and η is an energy large compared with the ionization potentials of the atomic electrons, i.e. about 10^4 ev. Except for extremely short distances x, collisions for which $E'<\eta$ occur so frequently that fluctuations due to them may be neglected and we may assume that they produce a constant rate of decline in the energy of the particle. (Unless otherwise specified, the term "particle" always refers to the ionizing particle.) This assumption will be made in calculations throughout this thesis; terms which are $O(\eta)$ will accordingly be neglected in all final results. As a consequence, the numerical results obtained are valid only when $P(E,x)$ is spread over a range of E values large compared with the highest atomic resonance level. When x is so small that this condition is not satisfied, the calculated curves will be too narrow, the fluctuation too small. At high energies ($E'_m >> \eta$), the condition is that ξ (Eq.q.4,2.5) be greater than the highest resonance level. This condition fails only when x is a very small fraction of the total range, when the curve is practically a single scattering curve. At low energies ($E'_m \lesssim \eta$), the condition of validity may fail for larger values of x; in this case the curve will be approximately Gaussian, and its moments can be calculated from the formulas to be given in Section 12, or in Chapter IV, when the moments of the collision spectrum χ are computed by the methods sketched in Chapter II, Section 5.

The term $-\beta^2 E'/E'_m$ in 8.1 is important only for close collisions where E' is an appreciable fraction of E'_m. When $\beta^2 << 1$, this term is always negligible; the curve $\chi(E')$ then falls abruptly to zero at $E' = E'_m$. When $\beta^2 \doteq 1$, this term just cancels the first term at $E' = E'_m$; $\chi(E')$ then strikes the axis at a small angle. Hereafter, when a particular value of β^2 is specified in referring to a curve (e.g. "$\beta^2 = 0$", "$\beta^2 = 1$" in Fig. 1d), it signifies that this value of β^2 was used in the second term in 8.1.

9. Derivation of the Fundamental Equation.

In deriving the fundamental equation for $P(E, x)$, it will be convenient to consider a very large number n of particles traversing a braking medium characterized by the collision spectrum $\chi(E', E, x)$, where, as indicated, χ may depend upon the path length x in the medium, and on the energy E of the particles. The total energy lost by a particle is $\mathcal{E}=E^0-E$, where E^0 is the energy of the particle at $x=0$, E its energy at x. For a given E^0, χ may be written as $\chi(E',\mathcal{E},x)$, and this form will be used throughout unless otherwise stated. $NP(\mathcal{E},x)d\mathcal{E}$ represents the number of particles which reach the point x having lost an amount of energy between $\mathcal{E}$ and $\mathcal{E}+d\mathcal{E}$. The number of these particles which suffer another collision in going an additional distance dx is:

$$-\left(\frac{\partial NP(\mathcal{E},x)}{\partial x}\right)_{out} d\mathcal{E}\,dx = NP(\mathcal{E},x)\,d\mathcal{E}\,dx\int_0^\infty \chi(E',\mathcal{E},x)\,dE' \tag{9.1}$$

The notation "$-\left(\frac{\partial NP}{\partial x}\right)_{out}$" is intended to express the fact that these particles are scattered out of the energy range $\mathcal{E}$ to $\mathcal{E}+d\mathcal{E}$; this represents a decrease in $NP(\mathcal{E})d\mathcal{E}$. The number of particles scattered into the energy range $\mathcal{E}$ to $\mathcal{E}+d\mathcal{E}$ in going a distance dx will be:

$$\left(\frac{\partial NP(\mathcal{E},x)}{\partial x}\right)_{in} d\mathcal{E}\,dx = d\mathcal{E}\,dx\int_0^\infty NP(\mathcal{E}-E',x)\chi(E',\mathcal{E}-E',x)\,dE' \tag{9.2}$$

The total change in $NP(\mathcal{E},x)d\mathcal{E}$ in going a distance dx is the sum $\left(\frac{\partial NP}{\partial x}\right)_{in} d\mathcal{E}\,dx + \left(\frac{\partial NP}{\partial x}\right)_{out} d\mathcal{E}\,dx$, and we have, dropping the constant factor $N\,d\mathcal{E}\,dx$:

$$\frac{\partial P(\mathcal{E},x)}{\partial x} = \int_0^\infty \left[P(\mathcal{E}-E',x)\chi(E',\mathcal{E}-E',x) - P(\mathcal{E},x)\chi(E',\mathcal{E},x)\right]dE' \tag{9.3}$$

This is the fundamental equation governing $P(\mathcal{E},x)$. It is essentially the Boltzmann equation of continuity. It can be written in a different form by changing the variable of integration in the first integral:

$$\frac{\partial P(\mathcal{E},x)}{\partial x} = \int_0^{\mathcal{E}} P(\mathcal{E}',x)\chi(\mathcal{E}-\mathcal{E}',\mathcal{E}',x)\,d\mathcal{E}' - P(\mathcal{E},x)\int_0^\infty \chi(E',\mathcal{E},x)\,dE' \tag{9.4}$$

(9.3 and 9.4 are equivalent provided $P(\mathcal{E},x)$ vanishes for $\mathcal{E}<0$, as will be the case for $x \geq 0$. Otherwise the lower limit in the first integral in 9.4 should be $-\infty$.)

A formal generalization of this equation which is sometimes convenient can be made if we suppose that, in addition to the finite energy losses E' due to collisions, the particle also suffers a continuous loss in energy according to:

$$\frac{d\mathcal{E}}{dx} = X(\mathcal{E},x) \qquad 9.5$$

where X is a known function. (X may for example be the function $\int_0^{\eta} E'\chi(E')dE'$ in 8.1, if we set $\chi = 0$ for $E' < \eta$.) We can always include the effect of X in the function χ by adding to χ the term:

$$\chi_X(E';\mathcal{E},x) = \frac{\delta(E')}{E'} X(\mathcal{E},x) \qquad 9.6$$

Such a term in χ merely adds the term X to the mean rate of energy loss $\int_0^\infty E'\chi(E';\mathcal{E},x)dE'$, but contributes no additional fluctuations. Adding the term χ_X to χ in 9.3 adds to the right side of 9.3 the term:

$$\int_0^\infty \left[P(\mathcal{E}-E',x)\frac{\delta(E')}{E'}X(\mathcal{E}-E',x) - P(\mathcal{E},x)\frac{\delta(E')}{E'}X(\mathcal{E},x)\right]dE'$$

$$= \int_0^\infty \frac{\delta(E')}{E'}\left[0 - E'\frac{\partial(PX)}{\partial\mathcal{E}} + \frac{1}{2}E'^2\frac{\partial^2(PX)}{\partial\mathcal{E}^2} + \cdots\right]dE'$$

$$= -\frac{\partial(PX)}{\partial\mathcal{E}}(\mathcal{E},x)$$

Thus we can include the effect of 9.5 by modifying 9.3, 9.4 as follows:

$$\frac{\partial P}{\partial x} = \int_0^\infty [P(\varepsilon-\varepsilon', x)\chi(\varepsilon', \varepsilon-\varepsilon', x) - P(\varepsilon, x)\chi(\varepsilon', \varepsilon, x)]\, d\varepsilon' - \frac{\partial}{\partial \varepsilon}(P\mathcal{I}) \qquad 9.7$$

$$\frac{\partial P}{\partial x} = \int_0^\varepsilon P(\varepsilon', x)\chi(\varepsilon-\varepsilon', \varepsilon', x)\, d\varepsilon' - P\int_0^\infty \gamma'(\varepsilon', \varepsilon, x)\, d\varepsilon' - \frac{\partial}{\partial \varepsilon}(P\mathcal{I}) \qquad 9.8$$

where it is understood that unless otherwise specified, all functions are evaluated at ε, x ; this convention will be followed hereafter.

It may be objected that the addition of $\chi_{\mathcal{I}}$ to χ makes the right side of 9.1 infinite, so that the use of 9.3 in deriving 9.7 is unjustified. Another derivation of 9.7 will therefore be given, where for simplicity we assume that $\chi = \chi_{\mathcal{I}}$, so that 9.5 represents the only energy loss. Then we will obviously have:

$$P(\varepsilon, x)\, d\varepsilon = P(\varepsilon+\Delta\varepsilon, x+\Delta x)\, d(\varepsilon+\Delta\varepsilon) \qquad 9.9$$

where Δx and $\Delta\varepsilon$ are related by 9.5:

$$\Delta\varepsilon = \mathcal{I}(\varepsilon, x)\,\Delta x + O((\Delta x)^2)$$

$$9.10$$

$$d(\Delta\varepsilon) = \frac{\partial \mathcal{I}}{\partial \varepsilon}\Delta x\, d\varepsilon + O((\Delta x)^2)$$

Inserting 9.10 in 9.9, neglecting terms $O((\Delta x)^2)$:

$$P\,d\varepsilon = P\,d\varepsilon + P\frac{\partial \mathcal{I}}{\partial \varepsilon}\Delta x\, d\varepsilon + \mathcal{I}\frac{\partial P}{\partial \varepsilon}\Delta x\, d\varepsilon + \frac{\partial P}{\partial x}\Delta x\, d\varepsilon$$

or:

$$\frac{\partial P}{\partial x} = -\frac{\partial (P\mathfrak{X})}{\partial \mathcal{E}} \qquad 9.11$$

This is the fundamental equation for $P(\mathcal{E},x)$ when the particles suffer only a continuous loss of energy given by 9.5. It can readily be solved if the general solution of 9.5 is known. If $\mathfrak{X}$ is independent of x, 9.5 gives:

$$x - x_0 = \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}}{\mathfrak{X}(\mathcal{E})} \qquad 9.12$$

9.12 defines a transformation:

$$\mathcal{E} = \mathcal{E}(\mathcal{E}_0, x - x_0)$$

$$\mathcal{E}_0 = \mathcal{E}_0(\mathcal{E}, x - x_0) \qquad 9.13$$

We get from 9.12:

$$\frac{\partial \mathcal{E}_0}{\partial \mathcal{E}} = \frac{\mathfrak{X}(\mathcal{E}_0)}{\mathfrak{X}(\mathcal{E})} \qquad 9.14$$

Since the variables $\mathcal{E}, x$ and $\mathcal{E}_0, x_0$ for each particle are related by 9.13, we can write:

$$P(\mathcal{E},x)\,d\mathcal{E} = P(\mathcal{E}_0,x_0)\,d\mathcal{E}_0 = P(\mathcal{E}_0,x_0)\frac{\partial \mathcal{E}_0}{\partial \mathcal{E}}\,d\mathcal{E}$$

and by 9.14:

$$P(\mathcal{E},x) = \frac{\mathfrak{X}(\mathcal{E}_0)}{\mathfrak{X}(\mathcal{E})} P(\mathcal{E}_0,x_0) \qquad 9.15$$

where $\mathcal{E}_0$ is given by 9.12 as a function of $\mathcal{E}, x - x_0$. It is readily verified that 9.15 satisfies 9.11 if $\frac{\partial \mathfrak{X}}{\partial x} = 0$. 9.15 gives us the connection between $P(\mathcal{E},x)$ at x and at x_0 whenever there are only negligible fluctuations in the energy lost between x and x_0.

10. Solution by Laplace Transform when $\frac{\partial\mathcal{V}}{\partial\mathcal{E}} = 0$.

The Laplace transform $\widetilde{A}(s)$ of a function $A(\mathcal{E})$ is defined by:

$$\widetilde{A}(s) = \int_0^\infty e^{-s\mathcal{E}} A(\mathcal{E})\,d\mathcal{E} \qquad 10.1$$

Here s is any complex number for which the integral converges. We shall need the following theorems (22, pp.152,161,105):

I. If $C(\mathcal{E}) = \frac{\partial A(\mathcal{E})}{\partial \mathcal{E}}$, and if $\widetilde{C}(s)$ exists, then:

$$\widetilde{C}(s) = s\widetilde{A}(s) - A(0)$$

II. If $\widetilde{A}(s)$, $\widetilde{B}(s)$ converge absolutely, and if:

$$C(\mathcal{E}) = \int_0^{\mathcal{E}} A(\mathcal{E}')B(\mathcal{E}-\mathcal{E}')\,d\mathcal{E}'$$

then:

$$\widetilde{C}(s) = \widetilde{A}(s)\widetilde{B}(s)$$

III. If $A(\mathcal{E})$ is continuous and properly integrable in every finite interval $0 \leq \mathcal{E} \leq L$, and if $\widetilde{A}(s)$ converges absolutely for $\mathcal{R}_e(s) > \alpha$, then:

$$A(\mathcal{E}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\mathcal{E}}\widetilde{A}(s)\,ds \qquad (\mathcal{E} > 0,\ c > \alpha)$$

When χ and I do not depend upon ε, 9.8 becomes:

$$\frac{\partial P}{\partial x} = \int_0^{\varepsilon} P(\varepsilon';x)\chi(\varepsilon-\varepsilon';x)\,d\varepsilon' - P\int_0^{\infty}\chi(\varepsilon';x)\,d\varepsilon' - I\frac{\partial P}{\partial \varepsilon} \qquad 10.2$$

This is the form of the fundamental equation whose solution is to be studied in this chapter. Applying the Laplace transformation to 10.2, using the theorems I and II:

$$\frac{\partial \tilde{P}(s,x)}{\partial x} = \tilde{P}(s,x)\left(\tilde{\chi}(s,x) - \tilde{\chi}(0,x) - sI(x)\right) + I(x)P(0,x) \qquad 10.3$$

($\tilde{\chi}(0,x)$ replaces $\int_0^{\infty}\chi(\varepsilon';x)\,d\varepsilon'$ by 10.1.) If I does not vanish, then $P(0,x)$ vanishes for $x > 0$. In any case, the last term in 10.3 vanishes for $x > 0$, and the solution is ($x_0 > 0$):

$$\tilde{P}(s,x) = \tilde{P}(s,x_0)\,e^{\int_{x_0}^{x}[\tilde{\chi}(s,x) - \tilde{\chi}(0,x) - sI(x)]\,dx} \qquad 10.4$$

As $x_0 \rightarrow 0$, $P(\varepsilon,x_0) \rightarrow \delta(\varepsilon)$, and we have from 10.1:

$$\tilde{P}(s,0) = \int_0^{\infty} e^{-s\varepsilon}\delta(\varepsilon)\,d\varepsilon = 1 \qquad 10.5$$

Letting $x_0 \rightarrow 0$ in 10.4:

$$\tilde{P}(s,x) = e^{\int_0^{x}[\tilde{\chi}(s,x) - \tilde{\chi}(0,x) - sI(x)]\,dx} \qquad 10.6$$

Note that if $\chi = 0$, 10.6 reduces to:

$$\tilde{P}(s,x) = e^{-s\int_0^{x} I(x)\,dx} \qquad 10.7$$

which is the transform of:

$$P(\varepsilon,x) = \delta\left(\varepsilon - \int_0^{x} I(x)\,dx\right) \qquad 10.8$$

as it should be. In the general case, the inversion of 10.6 is (theorem III):

$$P(\varepsilon,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\varepsilon + \int_0^{x}[\tilde{\chi}(s,x) - \tilde{\chi}(0,x) - sI(x)]\,dx}\,ds \qquad 10.9$$

where the path of integration is extended to the right of all poles of the integrand. 10.9 is the general solution of 10.2. The evaluation of the integral will occupy the remainder of this chapter. For convenience, we will hereafter assume that χ and $\mathcal{X}$ are not functions of x:

$$P(\varepsilon,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\varepsilon + [\tilde{\chi}(s) - \tilde{\chi}(0) - s\mathcal{X}]x}\, ds \qquad 10.10$$

All of the general results derived from 10.10 can be immediately extended to 10.9 in an obvious way, by replacing terms like χx by $\int_0^x \chi(x)\,dx$ wherever they occur.

11. Landau's Solution.

We obtain the problem whose solution was given by Landau (3) if we set $E'_m = \infty$ in 8.1. Let us write 8.1 in the form:

$$\chi(E') = \begin{cases} \dfrac{2\mu_e}{\beta^2}\dfrac{1}{E'^2} & E' > \eta \\ 0 & E' < \eta \end{cases} \qquad 11.1$$

$$\mathcal{X} = \frac{2\mu_e}{\beta^2}\log \eta/\varepsilon$$

where η is to be regarded as infinitesimal, and β^2 and ε are to be considered constant. By 11.1 and 10.1:

$$\tilde{\chi}(s) - \tilde{\chi}(0) = \frac{2\mu_e}{\beta^2}\int_\eta^\infty \frac{e^{-sE'}-1}{E'^2}\,dE'$$

(integrating by parts): $$= \frac{2\mu_e}{\beta^2}\left[\frac{1-e^{-sE'}}{E'}\right]_\eta^\infty - \frac{2\mu_e s}{\beta^2}\int_\eta^\infty \frac{e^{-sE'}}{E'}\,dE'$$

(we now require $Re(s) > 0$): $$= -\frac{2\mu_e s}{\beta^2} + O(s\eta) - \frac{2\mu_e s}{\beta^2}\int_{s\eta}^\infty \frac{e^{-z}}{z}\,dz$$

$$= -\frac{2\mu_e s}{\beta^2} + O(s\eta) - \frac{2\mu_e s}{\beta^2}\left[\int_1^\infty \frac{e^{-z}}{z}dz + \int_0^1 \frac{e^{-z}-1}{z}dz + \int_{s\eta}^1 \frac{dz}{z} - \int_0^{s\eta} \frac{e^{-z}-1}{z}dz\right]$$

$$\tilde{\chi}(s) - \tilde{\chi}(0) = \frac{2\mu_e s}{\beta^2}\left[C - 1 - \log\frac{1}{s\eta}\right] + O(s\eta) \qquad 11.2$$

The constant C is given by (23,p.2):

$$C = \int_0^1 \frac{1-e^{-z}}{z}dz - \int_1^\infty \frac{e^{-z}}{z}dz = 0.577 \qquad 11.3$$

Using 11.2, 11.1, omitting terms $O(s\eta)$:

$$\tilde{\chi}(s) - \tilde{\chi}(0) - s\bar{\chi} = \frac{2\mu_e s}{\beta^2}\left[C - 1 + \log s\epsilon\right] \qquad 11.4$$

Let:

$$\xi = \frac{2\mu_e x}{\beta^2}$$

$$\Delta_L = \frac{\epsilon - \epsilon_p}{\xi}, \quad \epsilon_p = \xi\left(\log \xi/\epsilon + j\right) \qquad 11.5$$

$$u = \xi s$$

where j is a constant to be specified later. Substituting 11.4 and 11.5 in 10.10:

$$P(\epsilon, x) = \frac{1}{2\pi i \xi}\int_{c-i\infty}^{c+i\infty} e^{u\log u + (\Delta_L + C - 1 + j)u}\, du \qquad 11.6$$

The integral in 11.6 is a function of $(\Delta_L + C - 1 + j)$ only. According to Landau, it has its maximum at -0.05. If we require that the maximum occur at $\Delta_L = 0$, this gives, using 11.3:

$$j = 1 - C - 0.05 = 0.37 \qquad 11.7$$

The function ϕ is defined by:

$$\phi(\Delta_L) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{u\log u + (\Delta_L - 0.05)u}\,du \qquad 11.8$$

Graphs of this function and its integral are given in (3); they are plotted in Fig. 1a and 1b (marked "$\lambda_w = 1.48$"). In terms of ϕ, 11.6 can be written:

$$P(\varepsilon,x) = \frac{1}{\xi}\,\phi\left(\frac{\varepsilon-\varepsilon_p}{\xi}\right) \qquad 11.9$$

In deriving 11.9, three approximations have been made; we have let $\gamma \to 0$, and $E'_m \to \infty$, and have assumed that β^2 and ϵ are independent of ε. The first and third assumptions are made throughout this chapter and their significance is discussed elsewhere. The second assumption is generally the more restrictive. Let us suppose that $E'_m/\xi >> 1$. In the case $\beta^2 = 0$ (in the second term in 8.1), the effect of setting $E'_m = \infty$ is then to add a small spurious probability for collisions in which E' is greater than the true E'_m. The result is to reduce the height of the curve $P(\varepsilon,x)$ slightly for $\Delta_L < \frac{E'_m}{\xi} - 2$, and to add to the tail of the curve for $\Delta_L > E'_m/\xi - 2$. This effect can be easily compensated for, as will be discussed in more detail in Section 17. In the case $\beta^2 = 1$, the effect on the curve is somewhat greater, and sets in at $\Delta_L \gtrsim 0.1\, E'_m/\xi$, if we regard a 10% change in the curve as significant. A glance at Fig. 1a will show that the principal part of the curve lies between $\Delta_L = -2$ and $\Delta_L \sim 7$, so that when $E'_m/\xi >> 1$ is not satisfied, there will be a significant error in the curve even in the neighborhood of the peak. The condition on E'_m/ξ is more stringent at large values of β^2, but this is somewhat compensated for by the fact that the ratio E_m'/E_k is larger at higher energies. For $\beta^2 \ll 1$, $E_m'/E_k \doteq 4\mu_e/\mu$ or about 0.02, whereas at $\beta^2 = 0.9$, $E_m'/E_k = 8\mu_e/\mu$ or 0.04. As $\beta^2 \to 1$, $E_m'/E_k \to 1$.

12. Moments and Semi-invariants.

The moments of the collision spectrum are defined by:

$$\chi_1 = I + \int_0^\infty E'\chi(E')\,dE' , \qquad \chi_n = \int_0^\infty E'^n \chi(E')\,dE' \tag{12.1}$$

By 10.1, the moments of any function $A(\varepsilon)$ are related to the derivatives of its Laplace transform at $s = 0$ by:

$$A_n = \int_0^\infty \varepsilon^n A(\varepsilon)\,d\varepsilon = (-1)^n\left[\frac{d^n}{ds^n}\tilde{A}(s)\right]_{s=0} \tag{12.2}$$

Expanding the exponent in the integrand of 10.10 in a Taylor series about $s = 0$, using 12.2:

$$P(\varepsilon,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s(\varepsilon-\chi_1 x) + \frac{s^2}{2!}\chi_2 x - \frac{s^3}{3!}\chi_3 x + \cdots}\,ds \tag{12.3}$$

Setting $s = -i\omega$, $c = 0$:

$$P(\varepsilon,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\varepsilon-\chi_1 x)(i\omega) + \frac{\chi_2 x}{2!}(i\omega)^2 + \frac{\chi_3 x}{3!}(i\omega)^3 + \cdots}\,d\omega \tag{12.4}$$

The Fourier transform of a distribution function $P(\varepsilon)$ is called its "characteristic function" (25,p.89):

$$\Pi(\omega) = \int_{-\infty}^{\infty} e^{i\omega\varepsilon} P(\varepsilon)\,d\varepsilon \tag{12.5}$$

The inverse transformation is:

$$P(\varepsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\varepsilon}\,\Pi(\omega)\,d\omega \tag{12.6}$$

By 12.6 and 12.4, the characteristic function of the distribution $P(\varepsilon,x)$ is:

$$\Pi(\omega,x) = \tilde{P}(-i\omega,x) = e^{\chi_1 x(i\omega) + \frac{\chi_2 x}{2!}(i\omega)^2 + \frac{\chi_3 x}{3!}(i\omega)^3 + \cdots} \qquad 12.7$$

The semi-invariants, introduced by Thiele (24), (25,p.186), of the distribution $P(\varepsilon)$ are defined as the coefficients K_ν in the series:

$$\log \Pi(\omega) = \sum_{\nu=1}^{\infty} \frac{K_\nu}{\nu!}(i\omega)^\nu \qquad 12.8$$

The identity of the semi-invariants K_ν of the energy loss distribution with the products $x\chi_\nu$ of the distance with the moments of the collision spectrum is obvious from 12.7 and 12.8. In case the collision spectrum depends on x, the relation is:

$$K_\nu = \int_0^x \chi_\nu(x)\,dx \qquad 12.9$$

This relation between the collision spectrum and the energy loss distribution is given essentially (i.e. except for terminology) in Cramèr's discussion of the general homogeneous random process in (19). It can also be proved by using the additivity property of the semi-invariants. If $P_1(\varepsilon_1)$, $P_2(\varepsilon_2)$ are the distributions of two independent random variables ε_1, ε_2, then the characteristic function of the distribution $P_{1+2}(\varepsilon_1+\varepsilon_2)$ of their sum is given by (25,p.188):

$$\Pi_{1+2}(\omega) = \Pi_1(\omega)\cdot\Pi_2(\omega) \qquad 12.10$$

From this and 12.8, it follows that:

$$K_{(1+2),\nu} = K_{(1),\nu} + K_{(2),\nu} \qquad 12.11$$

The energy loss distribution $P(\varepsilon, dx)$ when the particle goes an infinitesimal distance dx is the single collision distribution (it is assumed for simplicity that $I = 0$):

$$P(\varepsilon, dx) \doteq \left(1 - dx \int_0^\infty \chi(\varepsilon')\, d\varepsilon'\right) \delta(\varepsilon) + \chi(\varepsilon)\, dx \qquad 12.12$$

The Fourier transform is:

$$\begin{aligned} \Pi(\omega, dx) &\doteq 1 - dx\, \tilde{\chi}(0) + dx\, \tilde{\chi}(-i\omega) \\ &\doteq e^{[\tilde{\chi}(-i\omega) - \tilde{\chi}(0)]dx} \\ &\doteq e^{\chi_1 dx (i\omega) + \frac{\chi_2 dx}{2!}(i\omega)^2 + \frac{\chi_3 dx}{3!}(i\omega)^3 + \cdots} \end{aligned} \qquad 12.13$$

So the semi-invariants of the distribution $P(\varepsilon, dx)$ are:

$$\chi_{(dx),\nu} = \chi_\nu\, dx \qquad 12.14$$

By 12.11 and 12.14, we have immediately 12.9 for the semi-invariants of the distribution $P(\varepsilon, x)$ of the total energy lost by a particle in going the distance x.

The relations between the semi-invariants and the moments of $P(\varepsilon, x)$ can be obtained by applying 12.2 to its Laplace transform (from 12.3):

$$\tilde{P}(s, x) = e^{-s\chi_1 x + \frac{s^2}{2!}\chi_2 x - \frac{s^3}{3!}\chi_3 x + \cdots} \qquad 12.15$$

The result is, for the first few moments:

$$\begin{aligned} \bar{\varepsilon} &= \chi_1 x \\ \overline{\varepsilon^2} &= \chi_2 x + (\chi_1 x)^2 \end{aligned} \qquad 12.16$$

$$\overline{\mathcal{E}^3} = \varkappa_3 x + 3(\varkappa_2 x)(\varkappa_1 x) + (\varkappa_1 x)^3$$

$$\overline{\mathcal{E}^4} = \varkappa_4 x + 4(\varkappa_3 x)(\varkappa_1 x) + 3(\varkappa_2 x)^2 + 6(\varkappa_2 x)(\varkappa_1 x)^2 + (\varkappa_1 x)^4$$

etc.

The relations become simpler in terms of the central moments:

$$\begin{aligned}
\overline{(\Delta\mathcal{E})^2} &= \overline{(\mathcal{E}-\bar{\mathcal{E}})^2} = \varkappa_2 x \\
\overline{(\Delta\mathcal{E})^3} &= \overline{(\mathcal{E}-\bar{\mathcal{E}})^3} = \varkappa_3 x \\
\overline{(\Delta\mathcal{E})^4} &= \overline{(\mathcal{E}-\bar{\mathcal{E}})^4} = \varkappa_4 x + 3(\varkappa_2 x)^2 \\
\overline{(\Delta\mathcal{E})^5} &= \overline{(\mathcal{E}-\bar{\mathcal{E}})^5} = \varkappa_5 x + 10(\varkappa_3 x)(\varkappa_2 x) \\
\overline{(\Delta\mathcal{E})^6} &= \overline{(\mathcal{E}-\bar{\mathcal{E}})^6} = \varkappa_6 x + 15(\varkappa_4 x)(\varkappa_2 x) + 10(\varkappa_3 x)^2 + 15(\varkappa_2 x)^3
\end{aligned} \qquad 12.17$$

etc.

A general formula for the moment $\overline{(\Delta\mathcal{E})^n}$ can be derived from the formula:

$$S_n = e^{-z}\frac{d^n}{dy^n}e^z = \sum_{\sum_1^n jk_j = n} B_{k_1 k_2 \cdots k_n}(z')^{k_1}(z'')^{k_2}\cdots(z^{(n)})^{k_n} \qquad 12.18$$

where z is a function of y; the sum is taken over all products of derivatives of z containing a total of n primes; and $B_{k_1 k_2 \cdots k_n}$ is the number of different ways of carrying out n differentiations of e^z, differentiating only a single factor at a time, and arriving at the product $(z')^{k_1}(z'')^{k_2}\cdots(z^{(n)})^{k_n}e^z$. A rather intricate bit of combinatory analysis* results in the following formula for $B_{k_1 k_2 \cdots k_n}$:

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* Briefly, there are n operations to be performed. These can be permuted in $n!$ ways. For each factor $z^{(j)}$, the $j!$ permutations of the j differentiations of this factor do not give really different ways of carrying out the differentiation; hence the factors $(1!)^{k_1}\cdots(n!)^{k_n}$ in 12.19. In addition, permutations of the order of the k_j factors $z^{(j)}$ are not distinguishable; hence the factors $k_1!\cdots k_n!$ in 12.19.

$$B_{k_1 k_2 \cdots k_n} = \frac{\left(\sum_1^n j k_j\right)!}{(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}\, k_1!\, k_2! \cdots k_n!} \qquad 12.19$$

If we define:

$$S'_n = \sum_{\sum_2^n j k_j = n} B_{0 k_2 \cdots k_n} (z'')^{k_2} \cdots (z^{(n)})^{k_n} \qquad 12.20$$

then 12.18 can be expanded:

$$S_n = (z')^n + \binom{n}{2}(z')^{n-2} S'_2 + \cdots + \binom{n}{k}(z')^{n-k} S'_k + \cdots + n z' S'_{n-1} + S'_n \qquad 12.21$$

Using 12.2 and 12.15, and comparing 12.21 with the formula:

$$\overline{\epsilon^n} = \overline{(\bar{\epsilon} + \Delta\epsilon)^n} = \bar{\epsilon}^n + \binom{n}{2}\bar{\epsilon}^{n-2}\overline{(\Delta\epsilon)^2} + \cdots + \binom{n}{k}\bar{\epsilon}^{n-k}\overline{(\Delta\epsilon)^k} + \cdots$$

$$+ n\bar{\epsilon}\,\overline{(\Delta\epsilon)^{n-1}} + \overline{(\Delta\epsilon)^n} \qquad 12.22$$

noting that $\bar{\epsilon} = \chi_1 x$, we find that $\overline{(\Delta\epsilon)^n}$ is given by:

$$\overline{(\Delta\epsilon)^n} = \sum_{\sum_2^n j k_j = n} B_{0 k_2 \cdots k_n} (\chi_2 x)^{k_2} \cdots (\chi_n x)^{k_n} \qquad 12.23$$

When $x \to 0$, $\overline{\epsilon^k} \to \chi_k x$, i. e. the moments become those of the single collision distribution, as we should expect. When $x \to \infty$, $\overline{(\Delta\epsilon)^{2k}} \to [1\cdot 3\cdot 5 \cdots (2k-1)](\chi_2 x)^k$; these are the moments of a Gaussian distribution. As $x \to \infty$, the odd moments become unimportant compared with the even moments, as we see if we define the dimensionless moments:

$$\lambda_k = \frac{\overline{(\Delta\epsilon)^k}}{\left(\overline{(\Delta\epsilon)^2}\right)^{k/2}} \qquad 12.24$$

(I have been unable to find a general statistical terminology for the quantities λ_k, which I shall simply call the "dimensionless moments".) If we define the dimensionless moments of the collision spectrum:

$$\underline{\varkappa_k} = \frac{\varkappa_k}{\varkappa_2^{k/2}} \qquad 12.25$$

then, by 12.17 and 12.24:

$$\lambda_2 = 1$$

$$\lambda_3 = \underline{\varkappa_3}\, x^{-1/2}$$

$$\lambda_4 = \underline{\varkappa_4}\, x^{-1} + 3 \qquad 12.26$$

$$\lambda_5 = \underline{\varkappa_5}\, x^{-3/2} + 10\, \underline{\varkappa_3}\, x^{-1/2}$$

$$\lambda_6 = \underline{\varkappa_6}\, x^{-2} + (15\, \underline{\varkappa_4} + 10\, \underline{\varkappa_3}^2)\, x^{-1} + 15$$

etc.

It is convenient to have a symbol also for the (unweighted) r.m.s. fluctuation:

$$\sigma_u = \left(\overline{(\Delta E)^2}\right)^{1/2} \qquad 12.27$$

According to 12.17:

$$\sigma_u = (\varkappa_2 x)^{1/2} \qquad 12.28$$

The formulas 12.26 and 12.28 are of course only valid when $\varkappa(E')$ is independent of $\mathcal{E}$ and x. The generalization to the case where $\varkappa$ depends on x is obvious; the case where $\varkappa$ depends on $\mathcal{E}$ is more difficult and will be treated in the next chapter. As $x \to \infty$, $\lambda_{2k} \to 1 \cdot 3 \cdot 5 \cdots (2k-1)$, $\lambda_{2k+1} \to 0$. These are the moments of a Gaussian curve with unit second moment. Thus we have proved the central limit theorem of the theory of probability as applied to the present problem, in the special case where $\mathcal{X}$ is independent of x and $\mathcal{E}$. The theorem is not true in general; if $\mathcal{X}$ depends on x, we must place enough restrictions on the functions $\mathcal{X}_k(x)$ to insure that the contributions to $\int_0^x \mathcal{X}_2(x')dx'$ from large values of x' are sufficient to make the unwanted terms in 12.26 negligible compared with the "Gaussian" terms.

The moments of the distribution $P(\mathcal{E},x)$ can be used to expand $P(\mathcal{E},x)$ in terms of a set of orthogonal polynomials. Consider a set of polynomials $Q_k(\mathcal{E})$ orthonormal with respect to the weight function $\rho(\mathcal{E})$:

$$\int_0^\infty Q_k(\mathcal{E})\, Q_l(\mathcal{E})\, \rho(\mathcal{E})\, d\mathcal{E} = \delta_{kl} \qquad 12.29$$

Suppose $P(\mathcal{E},x)$ can be expanded in a series as follows:

$$P(\mathcal{E},x) = \rho(\mathcal{E}) \sum_{k=0}^{\infty} c_k(x)\, Q_k(\mathcal{E}) \qquad 12.30$$

Multiplying by $Q_l(\mathcal{E})$ and integrating, using 12.29:

$$c_l(x) = \int_0^\infty Q_l(\mathcal{E})\, P(\mathcal{E},x)\, d\mathcal{E} \qquad 12.31$$

The coefficients $c_k(x)$ are given by 12.31 in terms of the moments 12.16 of $P(\mathcal{E},x)$ if Q_k is a polynomial. It will be noticed that an attempt to expand according to:

$$P(\mathcal{E},x) = \sum_{k=0}^{\infty} c_k(x)\, Q_k(\mathcal{E}) \qquad 12.32$$

will lead to integrals for the coefficients $c_k(x)$ which cannnot be evaluated in terms of the moments alone unless $\rho(\mathcal{E})$ is a polynomial. (In this case the $\mathcal{E}$ interval must be finite.) By choosing a weight function $\rho(\mathcal{E})$ which approximates the distribution function $P(\mathcal{E},x)$, we may expect the first few terms of 12.30 to give a good approximation to $P(\mathcal{E},x)$, since the first term ($Q_0 = 1$) is already pretty close. The discussion of this paragraph applies equally well to an expansion in terms of polynomials $Q_k(\Delta\mathcal{E})$ using the central moments 12.17, in which case we would probably use the interval $-\infty < \Delta\mathcal{E} < \infty$. The only expansion of this type that will be used in this thesis is the Edgeworth or Gram-Charlier expansion using Hermite polynomials with a Gaussian weight function. The formulas will be derived in Section 14.

13. Digression on the Convergence of the Integral 10.10.

In the Landau limit, $E'_m = \infty$, the transform $\tilde{\chi}(s)$ does not converge unless $\mathcal{R}e(s) \geq 0$. Whenever $E'_m < \infty$, $\tilde{\chi}(s)$ obviously converges absolutely throughout the entire s-plane, since the integral is then extended only over the finite interval $0 \leq E' \leq E'_m$. In general, as will be seen from 12.2, there is an intimate connection between the behavior of $\tilde{A}(s)$ near $s = 0$ and the behavior of $A(\mathcal{E})$ for large $\mathcal{E}$; the existence of the moment A_n, $n = 0, 1, 2, \ldots$, guarantees the existence of the corresponding derivative of $\tilde{A}(s)$ at $s = 0$ and vice versa (e.g. 22,p.186). As $|s| \to \infty$ along a line parallel to the imaginary s-axis, or any path lying wholly to the right of such a line, $\tilde{\chi}(s) \to 0$. If $|s| \to \infty$ along such a path that $\mathcal{R}e(s) \to -\infty$, $|\tilde{\chi}(s)| \to \infty$. Along any path, the inversion integral:

$$\chi(E') = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sE'}\,\tilde{\chi}(s)\,ds \qquad 13.1$$

converges, provided $c > -\infty$ (22,p.104).

$x\tilde{\chi}(0)$ represents the average number of collisions suffered by the particle in going a distance x. This number will be very large for any appreciable distance x, but if it is finite, there is a finite probability ($e^{-x\tilde{\chi}(0)}$) that the particle will suffer no collision at all; thus $P(\mathcal{E},x)$ will contain a small singular term $e^{-x\tilde{\chi}(0)}\delta(\mathcal{E})$. (The term I in the collision spectrum is here to be supposed eliminated, either by changing variables from $\mathcal{E}$ to $\mathcal{E}-Ix$, or by choosing $\eta=\epsilon$ in 8.1 so that I vanishes.) With this singular term, the inverse transform of $\tilde{P}(s,x)$ will not really converge. As will be seen from 10.10, when $\tilde{\chi}(s)\to 0$, $\tilde{P}(s,x)\to e^{-x\tilde{\chi}(0)}$, which is a very small number, but not zero. To secure convergence we may define the distribution $R(\mathcal{E},x)$ of particles which suffer at least one collision:

$$R(\mathcal{E},x) = P(\mathcal{E},x) - e^{-x\tilde{\chi}(0)}\delta(\mathcal{E}) \qquad 13.2$$

The transform of $R(\mathcal{E},x)$ is:

$$\tilde{R}(s,x) = e^{-x\tilde{\chi}(0)}\left(e^{x\tilde{\chi}(s)}-1\right) \qquad 13.3$$

The function defined by 13.3 will have the same zeros and singular points as $\tilde{\chi}(s)$, and their inversion integrals will have the same convergence behavior. Thus if $\mathcal{E}_m'<\infty$, the inversion integral for $R(\mathcal{E},x)$ converges for any $c>-\infty$, and we infer that $R(\mathcal{E},x)$, and therefore $P(\mathcal{E},x)$ also, falls to zero faster than $e^{-\alpha\mathcal{E}}$ for any finite α as $\mathcal{E}\to\infty$; it can be shown that $P(\mathcal{E},x)$ falls to zero faster than $e^{-\mathcal{E}\log\mathcal{E}}$ if $\mathcal{E}_m'<\infty$. When $\mathcal{E}_m'=\infty$, it can be shown (3) that $P(\mathcal{E},x)$ is asymptotic to the single collision law:

$$P(\varepsilon,x) \underset{(\varepsilon\to\infty)}{\sim} \frac{2\mu_e x}{\beta^2}\frac{1}{\varepsilon^2} \qquad 13.4$$

14. The Edgeworth Expansion (26).*

Starting with 12.4 and splitting off the first two terms in the exponent, let us expand the remaining factor in a power series in $(i\omega)$:

$$P(\varepsilon,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\varepsilon-\chi_1 x)(i\omega)+\frac{1}{2}\chi_2 x(i\omega)^2}\left[1+A_3(i\omega)^3+A_4(i\omega)^4+\cdots\right]d\omega \qquad 14.1$$

where:

$$A_3 = \frac{1}{6}\chi_3 x$$

$$A_4 = \frac{1}{24}\chi_4 x$$

$$A_5 = \frac{1}{120}\chi_5 x$$

$$14.2$$

$$A_6 = \frac{1}{720}\chi_6 x + \frac{1}{72}(\chi_3 x)^2$$

$$A_7 = \frac{1}{5040}\chi_7 x + \frac{1}{144}(\chi_4 x)(\chi_3 x)$$

$$A_8 = \frac{1}{8!}\chi_8 x + \frac{1}{720}(\chi_5 x)(\chi_3 x) + \frac{1}{1152}(\chi_4 x)^2$$

etc.

In general, making use of 12.18, we find:

$$A_n = \frac{1}{n!}\sum_{\sum_3^n jk_j = n} B_{00k_3\ldots k_n}(\chi_3 x)^{k_3}\cdots(\chi_n x)^{k_n} \qquad 14.3$$

---

* The treatment in this section is essentially that of Cramèr (25,Ch.17).

where $B_{00k_3\cdots k_n}$ is given by 12.19. Let the variables θ, Δ_u be defined by:

$$\Delta_u = \frac{\mathcal{E}-\bar{\mathcal{E}}}{\sigma_u} \qquad \left(\bar{\mathcal{E}} = \chi_1 x\,,\ \sigma_u = (\chi_2 x)^{1/2}\right) \tag{14.4}$$

$$\theta = \omega\sigma_u \tag{14.5}$$

The subscript "u" stands for "Unweighted"; as will be seen later, Δ_u is the variable by which the energy loss is conveniently characterized when the unweighted curve parameters are used. In the next section, a corresponding weighted variable Δ_w will be introduced. For the remainder of the present section, we drop the subscript "u", as no other subscript will be needed. Using 14.4 and 14.5, 14.1 can be written:

$$P(\mathcal{E},x) = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} e^{-i\theta\Delta} e^{-\frac{1}{2}\theta^2}\left[1 - \underline{A_3}(-i\theta)^3 + \underline{A_4}(-i\theta)^4 - \cdots\right] d\theta \tag{14.6}$$

where:

$$\underline{A_k} = \frac{A_k}{(\chi_2 x)^{k/2}} = \frac{A_k}{\sigma^k} \tag{14.7}$$

Differentiating n times with respect to Δ the well known formula (25,p.99):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta\Delta} e^{-\frac{1}{2}\theta} d\theta = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\Delta^2} = \varphi(\Delta) \tag{14.8}$$

we obtain:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta\Delta} e^{-\frac{1}{2}\theta^2}(-i\theta)^n d\theta = \frac{(-1)^n}{\sqrt{2\pi}} e^{-\frac{1}{2}\Delta^2} H_n(\Delta) = \varphi^{(n)}(\Delta) \tag{14.9}$$

where $H_n(\Delta)$ is the nth Hermite polynomial:

$$H_n(\Delta) = \sum_{k_1+2k_2=n} (-1)^{k_2} B_{k_1 k_2 0\cdots 0}\,\Delta^{k_1} \tag{14.10}$$

The first few Hermite polynomials are:

$$H_0(\Delta) = 1$$

$$H_1(\Delta) = \Delta$$

$$H_2(\Delta) = \Delta^2 - 1$$

$$H_3(\Delta) = \Delta^3 - 3\Delta$$

$$H_4(\Delta) = \Delta^4 - 6\Delta^2 + 3$$

$$H_5(\Delta) = \Delta^5 - 10\Delta^3 + 15\Delta$$

$$H_6(\Delta) = \Delta^6 - 15\Delta^4 + 45\Delta^2 - 15$$

Integrating 14.6 term by term, using 14.9, we find:

$$\begin{aligned} P(\mathcal{E},x) &= \frac{1}{\sigma_u\sqrt{2\pi}} e^{-\frac{1}{2}\Delta_u^2}\left[1 + \underline{A_3} H_3(\Delta_u) + \underline{A_4} H_4(\Delta_u) + \cdots\right] \\ &= \frac{1}{\sigma_u}\left[\phi(\Delta_u) - \underline{A_3}\phi'''(\Delta_u) + \underline{A_4}\phi^{(4)}(\Delta_u) - \cdots\right] \end{aligned} \qquad 14.11$$

This is the Gram-Charlier series for $P(\mathcal{E},x)$ (27),(28),(29). It is easily integrated to obtain the series:

$$\int_{-\infty}^{\mathcal{E}} P(\mathcal{E},x)\,d\mathcal{E} = \Phi(\Delta_u) - \underline{A_3}\phi''(\Delta_u) + \underline{A_4}\phi^{(3)}(\Delta_u) - \cdots \qquad 14.12$$

where

$$\Phi(\Delta) = \int_{-\infty}^{\Delta} \phi(\Delta)\,d\Delta \qquad 14.13$$

The Gram-Charlier series is not quite satisfactory for the present purpose, as is clear from the expressions for the coefficients:

$$\underline{A_3} = \frac{1}{6}\underline{\chi_3}\, x^{-1/2}$$

$$\underline{A_4} = \frac{1}{24}\underline{\chi_4}\, x^{-1}$$

$$\underline{A_5} = \frac{1}{120}\underline{\chi_5}\, x^{-3/2}$$

$$\underline{A_6} = \frac{1}{720}\underline{\chi_6}\, x^{-2} + \frac{1}{72}\underline{\chi_3}^2 x^{-1} \qquad 14.14$$

$$\underline{A_7} = \frac{1}{5040}\underline{\chi_7}\, x^{-5/2} + \frac{1}{144}\underline{\chi_4}\,\underline{\chi_3}\, x^{-3/2}$$

$$\underline{A_8} = \frac{1}{8!}\underline{\chi_8}\, x^{-3} + \left(\frac{1}{720}\underline{\chi_5}\,\underline{\chi_3} + \frac{1}{1152}\underline{\chi_4}^2\right) x^{-2}$$

etc.

It will be noticed that the coefficients of successive terms do not involve uniformly decreasing powers of **x**. Since the asymptotic character of our series for large values of **x** is clearly suggested by the behavior of the coefficients, as well as by the considerations of Section 12, it is more reasonable to rearrange the terms in the order of decreasing powers of **x**:

$$\begin{aligned}
P(\varepsilon, x) = \frac{1}{\sigma_u}\Big\{ & \phi(\Delta_u) - x^{-1/2}\frac{1}{6}\underline{\chi_3}\,\phi^{(3)}(\Delta_u) \\
& + x^{-1}\left[\frac{1}{24}\underline{\chi_4}\,\phi^{(4)}(\Delta_u) + \frac{1}{72}\underline{\chi_3}^2\phi^{(6)}(\Delta_u)\right] \\
& - x^{-3/2}\left[\frac{\underline{\chi_5}}{120}\phi^{(5)} + \frac{\underline{\chi_4}\,\underline{\chi_3}}{144}\phi^{(7)} + \frac{\underline{\chi_3}^3}{1296}\phi^{(9)}\right] \\
& + x^{-2}\left[\frac{\underline{\chi_6}}{720}\phi^{(6)} + \left(\frac{\underline{\chi_5}\,\underline{\chi_3}}{720} + \frac{\underline{\chi_4}^2}{1152}\right)\phi^{(8)} + \frac{\underline{\chi_4}\,\underline{\chi_3}^2}{1728}\phi^{(10)} + \frac{\underline{\chi_3}^4}{31{,}104}\phi^{(12)}\right] \\
& - x^{-5/2}\left[\frac{\underline{\chi_7}}{5040}\phi^{(7)} + \left(\frac{\underline{\chi_5}\,\underline{\chi_4}}{2880} + \frac{\underline{\chi_6}\,\underline{\chi_3}}{4320}\right)\phi^{(9)} + \left(\frac{\underline{\chi_5}\,\underline{\chi_3}^2}{8640} + \frac{\underline{\chi_4}^2\underline{\chi_3}}{6912}\right)\phi^{(11)}\right. \\
& \left. + \frac{\underline{\chi_4}\,\underline{\chi_3}^3}{31{,}104}\phi^{(13)} + \frac{\underline{\chi_3}^5}{933{,}120}\phi^{(15)}\right] + \cdots \Big\}
\end{aligned} \qquad 14.15$$

The asymptotic properties of this series have been investigated by Cramèr (19,p.98), who has shown that, under fairly general conditions, the integral of the series 14.15 really gives an asymptotic expansion of $\int_0^{\mathcal{E}} P(\mathcal{E},x)\,d\mathcal{E}$ in powers of $x^{-1/2}$, with a remainder of the form:

$$|R_n(\mathcal{E},x)| < \frac{M}{x^{n/2}} \qquad 14.16$$

Where $x^{-n/2}$ is the order of the first term dropped, and where M is independent of x and $\mathcal{E}$, but depends in general on n and on the spectrum $\chi(\mathcal{E}')$. It is ordinarily unnecessary to go beyond the terms of order x^{-1}. The labor of calculation rapidly becomes prohibitive beyond this point; and it is the present author's experience in applications to the sort of problem treated in this thesis, that, due to the asymptotic character of 14.15, when x is sufficiently large for inclusion of the terms in $x^{-3/2}$ and smaller powers to improve the accuracy of the result, the terms up to x^{-1} already give a sufficiently good approximation.

The moments of the meson collision spectrum 8.1, neglecting terms $O(\eta)$ are:

$$\chi_1 = \frac{2\mu_e}{\beta^2}\left[\log\frac{E'_m}{\epsilon} - \beta^2\right] = \frac{4\mu_e}{\beta^2}\left[\log\left(\frac{2\mu_e}{I_z}\frac{w^2-1}{(1+0.01\gamma w)^{1/3}}\right) - \beta^2\right]$$

$$\chi_n = \frac{2\mu_e}{\beta^2}\frac{E'^{\,n-1}_m}{n-1}\left[1-\beta^2\left(\frac{n-1}{n}\right)\right] \qquad (n>1) \qquad 14.17$$

$$= (2\mu_e)^n \frac{w^2(w^2-1)^{n-2}}{n-1}\frac{1}{(1+0.01\gamma w)^{n-1}}\left[1-\beta^2\left(\frac{n-1}{n}\right)\right]$$

$$\underline{\chi_n} = \frac{1}{n-1}\left(\frac{w^2-1}{w}\right)^{n-2}\frac{1}{(1+0.01\gamma w)^{\frac{n}{2}-1}}\,\frac{1-\beta^2\left(\frac{n-1}{n}\right)}{\left(1-\frac{1}{2}\beta^2\right)^{n/2}} \qquad 14.18$$

where we have introduced the abbreviation:

$$W = \frac{\mathcal{E}}{\mu} = \frac{\mathcal{E}_k}{\mu} + 1 \qquad 14.19$$

We will also use the abbreviation:

$$W_k = \frac{\mathcal{E}_k}{\mu} = W - 1 \qquad 14.20$$

The formula 14.18 is simplified, and comparison with the formulas of Section 16 is facilitated, if we define the ratio:

$$C = \frac{\mathcal{E}'_m}{\xi} = \frac{(W^2-1)^2}{x W^2} \frac{1}{(1+0.01\gamma W)} \qquad 14.21$$

The quantities appearing as coefficients in the series 14.15 are:

$$\underline{\varkappa_n} \, x^{-(\frac{n}{2}-1)} = \frac{C^{\frac{n}{2}-1}}{n-1} \frac{1-\beta^2(\frac{n-1}{n})}{(1-\frac{1}{2}\beta^2)^{n/2}} \qquad 14.22$$

We also note that:

$$\sigma_u = (\varkappa_2 x)^{1/2} = \xi \, C^{1/2} (1-\tfrac{1}{2}\beta^2)^{1/2} \qquad 14.23$$

$$\lambda_3 = \underline{\varkappa_3} \, x^{-1/2} = \frac{1}{2} C^{1/2} \frac{1-\frac{2}{3}\beta^2}{(1-\frac{1}{2}\beta^2)^{3/2}} \qquad 14.24$$

The latter two formulas are of course valid only when the dependence of the collision spectrum on x and $\mathcal{E}$ can be neglected.

The Edgeworth expansion 14.15, when terms up to and including $O(x^{-1})$ are used, gives a pretty good approximation to $P(\mathcal{E},x)$ for the collision spectrum 8.1 as long as $\lambda_3 \lesssim 1$. This can be seen in Fig. 1c, where the curve marked "Edgeworth expansion" is to be compared with the correct curve, marked "$\lambda_w = 1$". The parameter λ_3, defined by 12.24, is called the "coefficient of skewness", or briefly, the "skewness", of the curve $P(\mathcal{E},x)$. It gives the magnitude of the first correction term to the initial Gaussian term, and is a convenient measure of the deviation of $P(\mathcal{E},x)$ from a Gaussian curve, at least when the deviation is not too great. The subscript "$_3$" will ordinarily be dropped, and "λ" without a subscript will be understood to mean the skewness λ_3. This is convenient because we will want to distinguish the unweighted skewness λ_u, given by 12.24 with $k=3$, from the weighted skewness λ_w to be defined in the next section.

In the limiting case of small values of x, when the Landau approximations are valid, the expansion 14.15 breaks down altogether. The moments χ_1, χ_2, . . . , of the collision spectrum 11.1 are all infinite. For large finite values of $E_m'/\mathcal{E}$, the moments exist, but their value is determined almost entirely by the behavior of $P(\mathcal{E},x)$ at large values of $\mathcal{E}$ near $\mathcal{E}_p+E_m'$. The expansion 14.15 then gives no information about the behavior of $P(\mathcal{E},x)$ near the peak $\mathcal{E}_p$. A somewhat better series for small values of x is to be developed in the next section.

15. The Weighted Edgeworth Expansion, Derivation.

Another method which gives an asymptotic formula in inverse powers of x for the integral 10.10 is the saddle-point method (30,p.455). A saddle point s_0 is chosen satisfying:

$$\left[\frac{\partial}{\partial s}\left[s\mathcal{E} + (\tilde{\chi}(s) - \tilde{\chi}(0) - sX)x\right]\right]_{s=s_0} = 0 \qquad 15.1$$

This gives:

$$\mathcal{E}/x = -\tilde{\mathcal{X}}'(s_0) + \mathcal{I} \qquad 15.2$$

It can be shown that for the value of $\mathcal{E}/x$ given by 15.2, the real part of the exponent in the integral 10.10, if plotted vertically above the horizontal complex s-plane, has a saddle-point, or pass, at the point s_0. If we choose the path of integration so that it passes over the saddle-point and down the valleys on either side to $c \pm i\infty$, (where the real part of the exponent becomes $-\infty$), the integrand will have a maximum at s_0 and will fall off exponentially on either side, so that the principal contribution to the integral comes from the neighborhood of s_0. It is not difficult to show that in the present case s_0 always lies on the real axis if $\mathcal{E}/x$ is real, and that the path up the valleys is perpendicular to the real axis at s_0. The usual saddle-point formula is obtained by using as path of integration a straight line through s_0 from $s_0 - i\infty$ to $s_0 + i\infty$; if the valley is reasonably straight, this path will coincide with the valley over the important part of its length. We define the generalized moments of the collision spectrum by:

$$\mathcal{X}_0(s) = \tilde{\mathcal{X}}(s) - \tilde{\mathcal{X}}(0) = \int_0^\infty (e^{-sE'} - 1)\, \mathcal{X}(E')\, dE'$$

$$\mathcal{X}_1(s) = -\tilde{\mathcal{X}}'(s) + \mathcal{I} = \mathcal{I} + \int_0^\infty e^{-sE'} E' \mathcal{X}(E')\, dE' \qquad 15.3$$

$$\mathcal{X}_n(s) = (-1)^n \tilde{\mathcal{X}}^{(n)}(s) = \int_0^\infty e^{-sE'} E'^n \mathcal{X}(E')\, dE' \quad (n \geq 2)$$

For $s = 0$, the generalized moments reduce to the ordinary moments 12.1. Using 15.3, we expand the exponent in 10.10 about $s = s_0$, setting $s = s_0 - i\omega$, to arrive at a formula analogous to 12.4:

$$P(\mathcal{E},x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{s_0\mathcal{E} + x\chi_0(s_0) - [\mathcal{E} - x\chi_1(s_0)](i\omega) + \frac{x\chi_2(s_0)}{2!}(i\omega)^2 + \frac{x\chi_3(s_0)}{3!}(i\omega)^3 + \cdots} d\omega \qquad 15.4$$

Expanding the integrand in 15.4, we get the analogue of 14.1:

$$P(\mathcal{E},x) = \frac{1}{2\pi} e^{x\chi_0(s_0) + s_0\mathcal{E}} \int_{-\infty}^{\infty} e^{-[\mathcal{E} - x\chi_1(s_0)](i\omega) - \frac{x\chi_2(s_0)}{2}\omega^2}\Big[1 + A_3(s_0)(i\omega)^3 + A_4(s_0)(i\omega)^4 + \cdots\Big] d\omega \qquad 15.5$$

where $A_k(s_0)$ is defined by the obvious generalization of 14.2,14.3. When $\mathcal{E}$ and x are related by 15.2, the linear term in ω in the exponent vanishes. Integrating term by term, the terms corresponding to odd powers of ω drop out, and we obtain:

$$P(x\chi_1(s_0),x) = \frac{1}{[2\pi x\chi_2(s_0)]^{1/2}} e^{x\chi_0(s_0) + s_0 x\chi_1(s_0)}\Big[1 + 3\underline{A}_4(s_0) - 15\underline{A}_6(s_0) + 105\underline{A}_8(s_0) - \cdots\Big] \qquad 15.6$$

$\underline{A}_k(s_0)$ is of course defined in analogy with 14.7. The terms in 15.6 should be rearranged in order of decreasing powers of x:

$$P(x\chi_1(s_0),x) = \frac{1}{[2\pi x\chi_2(s_0)]^{1/2}} e^{x[\chi_0(s_0) + s_0\chi_1(s_0)]}\Big\{1 + x^{-1}\Big[\frac{1}{8}\underline{\chi}_4(s_0) + \frac{5}{24}\underline{\chi}_3^2(s_0)\Big] + x^{-2}\Big[\frac{1}{48}\underline{\chi}_6(s_0) + \frac{7}{48}\underline{\chi}_5(s_0)\underline{\chi}_3(s_0) + \frac{35}{384}\underline{\chi}_4^2(s_0) + \frac{115}{192}\underline{\chi}_4(s_0)\underline{\chi}_3^2(s_0) + \frac{385}{1152}\underline{\chi}_3^4(s_0)\Big] + \cdots\Big\} \qquad 15.7$$

$\underline{\kappa}_k(s_0)$ is defined in analogy with 12.25:

$$\underline{\kappa}_k(s_0) = \frac{\kappa_k(s_0)}{[\kappa_2(s_0)]^{k/2}} \qquad 15.8$$

15.7 is the saddle-point formula for $P(\mathcal{E},x)$ at the point $\mathcal{E} = x\,\kappa_1(s_0)$. Its derivation is in close analogy with that of the Edgeworth expansion formula 14.15. The analogy is exact up to formula 15.5. The derivation then deviates somewhat from that of Section 14, in that formula 15.6, obtained by integrating 15.5 term by term, is specialized to a particular value of $\mathcal{E}$ given by 15.2. For a given saddle point s_0, and a given x, formula 15.7 represents just one point of the curve $P(\mathcal{E},x)$ vs $\mathcal{E}$. In order to use 15.7 to determine $P(\mathcal{E},x)$, we must choose, for each value of $\mathcal{E}$, an appropriate saddle point s_0. For $s_0 = 0$, 15.7 is identical with the Edgeworth formula evaluated at $\mathcal{E} = x\kappa_1 = \bar{\mathcal{E}}$, i.e. at $\Delta_u = 0$.

We could have proceeded in exact analogy with the Edgeworth derivation to obtain a formula analogous to 14.15. 15.5 can be rewritten in analogy with 14.6 by defining (analogous to 14.4 and 14.5):

$$\Delta = \frac{\mathcal{E} - x\,\kappa_1(s_0)}{[x\,\kappa_2(s_0)]^{1/2}}$$

15.9

$$\theta = \omega\,[x\,\kappa_2(s_0)]^{1/2}$$

Integrating term by term the resulting analogue of 14.6, we arrive at a formula, precisely analogous to the Gram-Charlier formula 14.11, which can be rewritten to correspond to the Edgeworth formula 14.15. The path of integration is here the same as in the saddle-point method, but the resulting formula is not specialized to a particular value of ε. If s_0 is the saddle-point for $\varepsilon = x\kappa_1(s_0)$, it should be near the saddle point for values of ε near $x\kappa_1(s_0)$, so that we should expect the generalized Edgeworth expansion to be a good approximation to $P(\varepsilon, x)$ for ε near $x\kappa_1(s_0)$, at least if the saddle-point method gives a good approximation at $\varepsilon = x\kappa_1(s_0)$. We are interested in $P(\varepsilon, x)$ near its maximum at ε_p. Let us choose the saddle-point s_0 corresponding to ε_p. (s_0 will then be a function of x.) From now on the subscript "$_0$" will be dropped, as we are through with the complex plane.

We define:

$$\begin{aligned} \varepsilon_p &= x\kappa_1(s) \\ \sigma_w &= [x\kappa_2(s)]^{1/2} \\ \lambda_w &= x^{-1/2}\underline{\kappa_3}(s) \\ \zeta_w &= x^{-1}\underline{\kappa_4}(s) \\ \Delta_w &= \frac{\varepsilon - \varepsilon_p}{\sigma_w} \end{aligned} \qquad 15.10$$

ε_p is here defined as a function of s. We intend to choose s so that ε_p thus defined becomes the abscissa of the peak of $P(\varepsilon, x)$; the formal development is valid for any choice of s. ζ_w is the weighted "coefficient of excess". The weighted Edgeworth expansion, analogous to 14.15, is, writing terms up to $O(x^{-1})$:

$$P(\varepsilon,x) = \frac{1}{\sigma_w} e^{x\chi_0(s)+s\varepsilon}\Big\{\phi(\Delta_w) - \frac{\lambda_w}{6}\phi^{(3)}(\Delta_w) + \frac{\zeta_w}{24}\phi^{(4)}(\Delta_w) + \frac{\lambda_w^2}{72}\phi^{(6)}(\Delta_w) + \cdots\Big\} \qquad 15.11$$

Let the numbers a, b, d, be defined by:

$$\sigma_w = b\xi = \frac{2\mu_e x}{\beta^2} b$$

$$s = \frac{1}{a\xi} = \frac{b}{a\sigma_w} \qquad 15.12$$

$$\chi_0(s) = -s\chi_1(s) - d/x$$

Then 15.11 becomes:

$$P(\varepsilon,x) = \frac{e^{-d+\frac{b}{a}\Delta_w}}{\sigma_w}\Big\{\phi(\Delta_w) - \frac{\lambda_w}{6}\phi^{(3)}(\Delta_w) + \frac{\zeta_w}{24}\phi^{(4)}(\Delta_w) + \frac{\lambda_w^2}{72}\phi^{(6)}(\Delta_w) + \cdots\Big\} \qquad 15.13$$

The subscript "$_w$" will be dropped for the remainder of this section and the next, whenever no confusion with the unweighted parameters can arise.

We now specify the saddle-point s (i.e., by 15.12, a) so that ε_p really is the abscissa of the peak, i.e., so that the curve has a maximum at $\Delta = 0$. Let 15.13 be written as:

$$P(\varepsilon,x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-d+\frac{b}{a}\Delta-\frac{1}{2}\Delta^2}\Big[1 + A + B\Delta + O(\Delta^2)\Big] \qquad 15.14$$

Setting the derivative of the right hand side equal to zero for $\Delta = 0$, we obtain the condition for a:

$$a = \frac{b(1+A)}{B} \qquad 15.15$$

15.15 is not as simple as it looks, since A, B, and b are also functions of a. λ, ζ, etc., and b and d must be calculated as functions of a from 15.10 and 15.12. 15.15 is then solved for a and the result substituted in 15.13 to obtain $P(\varepsilon,x)$. We thus locate the peak ε_p of the curve and obtain a formula for $P(\varepsilon,x)$ near ε_p with little more calculation than is required by the ordinary saddle-point method to find a single point on the curve.

For numerical computation of $P(\varepsilon,x)$, using 15.13 and tables of ϕ, ϕ', etc., it is convenient to shift the origin of Δ, using the formula:

$$H_n(\Delta) = \sum_{k=0}^{n} \binom{n}{k} \Delta_0^{\,n-k} H_k(\Delta-\Delta_0) \qquad 15.16$$

This formula is easily derived from the formula for the Hermite polynomials in terms of their generating function :*

$$e^{\Delta t-\frac{1}{2}t^2} = H_0(\Delta) + tH_1(\Delta) + \frac{1}{2!}t^2H_2(\Delta) + \frac{1}{3!}t^3H_3(\Delta) + \cdots \qquad 15.17$$

Using 15.14 to write the terms up to $O(x^{-1})$ in 15.13 in terms of:

$$\Delta' = \Delta - b/a \qquad 15.18$$

we eliminate the exponential factor $e^{\frac{b}{a}\Delta}$:

$$P(\varepsilon,x) \doteq \frac{1}{\sigma} e^{-d+\frac{1}{2}\frac{b^2}{a^2}} \sum_{k=0}^{6} (-1)^k A'_k\, \phi^{(k)}(\Delta') \qquad 15.19$$

---

* The formula 15.17 is given in almost any reference on Hermite polynomials (e.g. 31, p.181). The author has not seen the formula 15.16 in the literature, but it must be well known. The derivation consists in substituting $(\Delta - \Delta_0)$ for Δ in 15.17, multiplying by $e^{\Delta_0 t}$, and collecting terms in the same power of t on the right hand side.

where the coefficients are given by:

$$A_0' = 1 + \frac{b^3}{a^3}\frac{\lambda}{6} + \frac{b^4}{a^4}\frac{\zeta}{24} + \frac{b^6}{a^6}\frac{\lambda^2}{72}$$

$$A_1' = \frac{b^2}{a^2}\frac{\lambda}{2} + \frac{b^3}{a^3}\frac{\zeta}{6} + \frac{b^5}{a^5}\frac{\lambda^2}{12}$$

$$A_2' = \frac{b}{a}\frac{\lambda}{2} + \frac{b^2}{a^2}\frac{\zeta}{4} + \frac{b^4}{a^4}\frac{5\lambda^2}{24}$$

$$A_3' = \frac{\lambda}{6} + \frac{b}{a}\frac{\zeta}{6} + \frac{b^3}{a^3}\frac{5\lambda^2}{18} \qquad 15.20$$

$$A_4' = \frac{\zeta}{24} + \frac{b^2}{a^2}\frac{5\lambda^2}{24}$$

$$A_5' = \frac{b}{a}\frac{\lambda^2}{12}$$

$$A_6' = \frac{\lambda^2}{72}$$

The integral of 15.19 is of course:

$$\int_{-\infty}^{\varepsilon} P(\varepsilon,x)\,d\varepsilon = e^{-d+\frac{1}{2}\frac{b^2}{a^2}}\left[A_0'\,\Phi(\Delta') + \sum_{k=1}^{6}(-1)^k A_k'\,\varphi^{(k-1)}(\Delta')\right] \qquad 15.21$$

Before applying the weighted Edgeworth expansion, some general considerations may be in order. The manner of its derivation leads to the belief that it will represent the curve best in the neighborhood of the peak ε_p Comparing 15.13 with 14.15, using 15.12, we see that the weighted expansion amounts to an ordinary Edgeworth expansion of the function:

$$P_w(\varepsilon,x) = e^{d-\frac{\varepsilon-\varepsilon_p}{a\xi}}P(\varepsilon,x) \qquad 15.22$$

We may define the quantities $a\xi$ and d without reference to the expansion 15.13 by requiring:

$$\int_0^{\infty} (\mathcal{E}-\mathcal{E}_p)\, e^{-\frac{\mathcal{E}-\mathcal{E}_p}{a\xi}} P(\mathcal{E},x)\, d\mathcal{E} = 0 \qquad 15.23$$

$$\int_0^{\infty} e^{d-\frac{\mathcal{E}-\mathcal{E}_p}{a\xi}} P(\mathcal{E},x)\, d\mathcal{E} = 1 \qquad 15.24$$

To the extent that 15.13 approximates $P(\mathcal{E},x)$, the values of $a\xi$ and d defined by 15.23 and 15.24 will agree with those defined by 15.12 and 15.15; if we substitute 15.13 for $P(\mathcal{E},x)$ in 15.23 and 15.24, they reduce to identities. Thus $P_w(\mathcal{E},x)$ is obtained by applying a weight factor which reduces the tail of the curve on the positive side of the peak, and raises the tail on the negative side, enough to make the mean value of $\mathcal{E}$ coincide with the peak value $\mathcal{E}_p$. One would expect the more symmetric curve $P_w(\mathcal{E},x)$ to be more easily represented by an Edgeworth series than the lopsided unweighted curve. If we assume that the error in the (ordinary Edgeworth) series for P_w is of roughly the same order of magnitude over the important part of the $\mathcal{E}$ scale; as is the case with the ordinary Edgeworth expansion of the distribution $P(\mathcal{E},x)$ (see Eq.14.16), then the error in the function:

$$P(\mathcal{E},x) = e^{-d+\frac{\mathcal{E}-\mathcal{E}_p}{a\xi}} P_w(\mathcal{E},x) \qquad 15.25$$

will increase exponentially toward positive values of $(\mathcal{E}-\mathcal{E}_p)$, and decrease exponentially toward negative values. This is precisely the sort of behavior that we are going to find, as a glance at Figures 1a and 1b will show. (The solid curves in Figures 1a and 1b are the correct distribution curves for the case when Landau's solution is valid. The dotted and dashed curves marked "1st approximation" and "2nd approximation" are obtained for the same case using the weighted Edgeworth expansion. These curves will be discussed in more detail in the next section.) Having defined d and $a\xi$ by 15.23 and 15.24, we can define the weighted parameters λ_w, σ_w by:

$$\sigma_w^2 = \int_0^\infty (\varepsilon-\varepsilon_p)^2 P_w(\varepsilon,x)\,d\varepsilon \qquad 15.26$$

$$\lambda_w = \frac{1}{\sigma_w^3}\int_0^\infty (\varepsilon-\varepsilon_p)^3 P_w(\varepsilon,x)\,d\varepsilon \qquad 15.27$$

One may ask how it is possible that we have found three different asymptotic expansions, 14.15, 15.7, 15.13, for the same function. The answer is that $P(\varepsilon,x)$ depends on two variables, and in order to get an expression asymptotic in one variable, say x, we must specify some relation:

$$a(\varepsilon,x) = c \qquad 15.28$$

which is to be held fixed as x varies. For a fixed value of c, $P(\varepsilon,x)$ is a function of x only, and we may ask for the asymptotic expansion of this function as $x \to \infty$. Different choices of the function $a(\varepsilon,x)$ will result in different functional dependences of $P(\varepsilon,x)$ on x when c is fixed, and therefore in different asymptotic formulas. 15.7, 14.15, and 15.13 correspond to the choices:

Saddle-point: $$a(\varepsilon,x) = \varepsilon/x \qquad 15.29$$

Edgeworth: $$\mathcal{A}(\mathcal{E},x) = \frac{\mathcal{E}-\bar{\mathcal{E}}(x)}{\sigma_u(x)} = \Delta_u \qquad 15.30$$

Weighted Edgeworth: $$\mathcal{A}(\mathcal{E},x) = \frac{\mathcal{E}-\mathcal{E}_p(x)}{\sigma_w(x)} = \Delta_w \qquad 15.31$$

When $\mathcal{A}(\mathcal{E},x)$ is chosen, $P(\mathcal{E},x)$ can be considered by 15.28, as a function of C and x. Presumably the best choice of $\mathcal{A}(\mathcal{E},x)$ would be that function which makes $P(\mathcal{E},x)$, plotted as a function of C, vary as little and as smoothly as possible with x, as then the asymptotic formula should be good over a wide range of values of x. The mean value $\bar{\mathcal{E}}$ varies linearly with x. The peak value $\mathcal{E}_p$ varies as $x\log x$ for small values of x, but becomes linear in x as it approaches $\bar{\mathcal{E}}$ for large values of x. σ_u is proportional to $x^{1/2}$; while σ_w, which better characterizes the width of the curve near the peak, increases in proportion to x for small values of x, (as we shall see later), and finally becomes proportional to $x^{1/2}$ as it approaches σ_u for large values of x. Thus if we plot $P(\mathcal{E},x)$ vs $C = \mathcal{E}/x$, then as x increases from zero to infinity, the abscissa $\mathcal{E}_p/x$ of the peak will at first increase in proportion to $\log x$, and ultimately come to a stop; while the width of the curve will at first be constant, and ultimately shrink in proportion to $x^{-1/2}$. If we plot P vs $C = \Delta_u$, the peak moves as in the previous case, while the width of the curve at first increases in proportion to $x^{1/2}$, and ultimately becomes constant. If we plot P vs $C = \Delta_w$, the peak remains fixed, and the width of the curve remains fixed; only the shape of the curve changes as the positive tail is reduced until the curve ultimately becomes Gaussian. The latter behavior is illustrated by Figure 2 (differential curve). On this basis, one might expect 15.31 to give the best asymptotic expansion.

16. The Weighted Edgeworth Expansion, Application.

We first compute the generalized moments 15.3 of the collision spectrum 8.1. The calculation of $\chi_0(s)$ and $\chi_1(s)$ uses some of the manipulations involved in the calculation of 11.4, with obvious modifications due to the finite value of E'_m. Otherwise the computation is perfectly straightforward, and only the result need be given here. (Terms $O(\eta)$ are neglected as usual):

$$\chi_0(s) = -s\chi_1(s) - \frac{2\mu_a s}{\beta^2} + \frac{2\mu_a}{\beta^2 E'_m}\Big[1 - e^{-sE'_m} - \beta^2\left(1 - e^{-sE'_m} + Ei(-sE'_m) - \log sE'_m - C\right)\Big] \qquad 16.1$$

$$\chi_1(s) = \frac{2\mu_a}{\beta^2}\left[\log\frac{1}{s\epsilon} - C + Ei(-sE'_m) - \beta^2\left(\frac{1-e^{-sE'_m}}{sE'_m}\right)\right]$$

$$= \chi_1(0) + \frac{2\mu_a}{\beta^2}\left[\log\frac{1}{sE'_m} - C + Ei(-sE'_m) - \beta^2\left(\frac{1-sE'_m-e^{-sE'_m}}{sE'_m}\right)\right] \qquad 16.2$$

The exponential integral is defined by (23,p.1):

$$-Ei(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \qquad 16.3$$

The terms multiplied by β^2 in the brackets in 16.1 and 16.2 come from the second term in the formula 8.1 for $\chi(E')$. (See the discussion following 8.1.) If we set $\beta^2=0$, i.e.,drop the "β^2" term in 8.1, we obtain for the higher generalized moments:

$$\chi_n^+(s) = \frac{2\mu_a}{\beta^2}\,\frac{(n-2)!}{s^{n-1}}\left[1-\sum_{k=0}^{n-2}\frac{1}{k!}(sE'_m)^k e^{-sE'_m}\right] \quad (n>1) \qquad 16.4$$

The general formula, including the β^2 term, is then:

$$\chi_n(s) = \chi_n^+(s) - \frac{\beta^2}{E'_m}\chi_{n+1}^+(s) \qquad 16.5$$

The formulas 16.2, 16.4, 16.5, reduce to 14.17 when $s = 0$. Formulas for $Ei(-x)$ for small and large values of x are useful (23,p.3):

$$-Ei(-x) = \log\frac{1}{\gamma x} + x - \frac{x^2}{4} + \cdots + \frac{(-1)^{n+1}}{n\,n!}x^n + \cdots \quad (\log\gamma = C) \qquad 16.6$$

$$-Ei(-x) = \frac{e^{-x}}{x}\left[1 - \frac{1!}{x} + \frac{2!}{x^2} - \cdots\right] \qquad |x| >> 1 \qquad 16.7$$

If we use the notations introduced by 15.10, 15.12, and 14.21, we can rewrite 16.1, and 16.2:

$$d = \frac{1}{a} - \frac{1-e^{-c/a}}{c}(1-\beta^2) - \frac{\beta^2}{c}\left[C + \log c/a - Ei(-c/a)\right] \qquad 16.8$$

$$\mathcal{E}_p = \xi\left[\log\frac{\xi}{\epsilon} + j\right] = \bar{\mathcal{E}} - \xi\left[\log c - j - \beta^2\right] \qquad 16.9$$

where:

$$j = -C + \log a + Ei(-c/a) - \beta^2\frac{a}{c}(1-e^{-c/a}) \qquad 16.10$$

16.4 may be written:

$$x\chi_n^+(s) = \xi^n a^{n-1}(n-2)!\left[1 - \sum_{k=0}^{n-2}\frac{1}{k!}\left(\frac{c}{a}\right)^k e^{-c/a}\right] \quad (n>1) \qquad 16.11$$

By 16.11, the quantities b, λ, ζ, (Eq.15.12, 15.10 become, in case $\beta^2 = 0$:

$$b^+ = a^{1/2}(1-e^{-c/a})^{1/2} \qquad 16.12$$

$$\lambda^+ = a^{1/2}\frac{1-(1+c/a)e^{-c/a}}{(1-e^{-c/a})^{3/2}} \qquad 16.13$$

$$\zeta^{+} = 2a\,\frac{1-\left(1+c/a+\frac{1}{2}c^2/a^2\right)e^{-c/a}}{\left(1-e^{-c/a}\right)^2} \qquad 16.14$$

The general formula for $\chi^{-(\frac{n}{2}-1)}\underline{\chi_n}^{+}(s)$ is easily obtained from 16.11, but we shall need only the special cases 16.13 and 16.14. The corresponding formulas for the case $\beta^2 \neq 0$ are, from 16.5, 16.12 - 16.14:

$$b = b^{+}\left(1-\frac{\beta^2}{c}b^{+}\lambda^{+}\right)^{1/2} \qquad 16.15$$

$$\lambda = \lambda^{+}\,\frac{1-\frac{\beta^2}{c}\frac{b^{+}\zeta^{+}}{\lambda^{+}}}{\left(1-\frac{\beta^2}{c}b^{+}\lambda^{+}\right)^{3/2}} \qquad 16.16$$

The corresponding formula for ζ will fortunately not be needed.

Let us first consider the case $c=\infty$ (Landau limit). Our weighted formula is asymptotic for $c\to 0$, and its convergence may therefore be expected to be worst for $c=\infty$. When $c\to\infty$, the above formulas become:

$$d = 1/a$$

$$j = -C + \log a$$

$$b = a^{1/2} \qquad 16.17$$

$$\lambda = a^{1/2}$$

$$\zeta = 2a$$

The distinction between $\beta^2 = 0$ and $\beta^2 \neq 0$ disappears. We see already that the peak ε_p as given by 16.9 has the Landau form; that the width of the curve as measured by $\sigma_w = b\xi$ is linear in ξ, in agreement with Landau's result; and that the shape of the curve is independent of ξ when we use the variable $\Delta_w = \frac{\varepsilon - \varepsilon_p}{b\xi}$, which is proportional to Landau's variable $\Delta_L = \frac{\varepsilon - \varepsilon_p}{\xi}$. The shape is constant because the dimensionless weighted semi-invariants $x^{-(\frac{n}{2}-1)}\underline{\kappa_n}(s)$ (of which λ, ζ are the first two) are independent of ξ. The essential point is that $x\kappa_n(s)$ (Eq.16.11) is proportional to ξ^n, and otherwise independent of x when c/a is very large. Note that none of these qualitative observations would be easy to draw from the unweighted formulas 14.17, 14.22, 14.23. It remains to determine the value of a and check the numerical agreement.

We have to decide how many terms of the asymptotic formula are significant. Once the number of terms is decided upon, the quantities A and B (Eq.15.14) are readily determined from 15.13 and the expression 14.10 for the Hermite polynomials. In the present case, 15.15 reduces to a simple algebraic equation for a. The smallest number of terms we can use is up to $O(x^{-1/2})$, i.e. up to terms in λ only. The term in $O(x^{-1})$ involves ζ, and higher order terms involve higher order semi-invariants and greater labor. The table below gives the resulting values of A, B, a, j, $\frac{1}{\xi}P(\varepsilon_p)$ when we keep terms up to the orders listed. The Landau values of j and $\frac{1}{\xi}P(\varepsilon_p)$ are given for comparison.

Convergence of Weighted Edgeworth Expansion when $E'_m = \infty$

Last term used	A	B	a	Change in a	j	$\frac{1}{\xi}P(\varepsilon_p)$	$b = \lambda_w$
$O(x^{-1/2})$	0	$\frac{1}{2}a^{1/2}$	2	----	0.116	0.172	1.414
$O(x^{-1})$	$\frac{a}{24}$	$\frac{1}{2}a^{1/2}$	2.18	0.18	0.202	0.187	1.478
$O(x^{-3/2})$	$\frac{a}{24}$	$\frac{a^{1/2}}{2} - \frac{a^{3/2}}{48}$	2.45	0.27	0.319	0.187	1.566
$O(x^{-2})$	$a/24 - 0.012a^2$	$\frac{a^{1/2}}{2} - \frac{a^{3/2}}{48}$	2.28	0.17	0.247	0.176	1.543
$O(x^{-5/2})$	$a/24 - 0.012a^2$	$\frac{a^{1/2}}{2} - \frac{a^{3/2}}{48} + 4.33a^{5/2}$	----	very great	----	----	----
Landau	----	----	----	----	0.37	0.179	?

The columns of the above table exhibit the typical behavior of asymptotic expansions; the first few terms give more or less converging results, and we finally come to a term which gives a large change. If we follow the often used and rarely justified procedure of stopping one step short of the last "converging" term,* and assuming the error less than the change produced by the first term dropped, we should use terms up to $O(x^{-3/2})$. We then would have:

$$j = 0.32 \pm 0.07$$

$$\frac{1}{\xi}P(\varepsilon_p) = 0.187 \pm 0.011$$

---

* In the present case, however, we are not simply adding successive additional terms to a series. Each time we add a term, the value of a changes, and therefore the whole series changes.

The agreement with the Landau values is excellent, even to the order of magnitude of the error. An error of 0.07 in j corresponds to a shift of the peak of the curve by 0.07 on the $\frac{\varepsilon-\varepsilon_p}{\xi}$ scale in Figure 1a; a shift of several times this amount would be barely perceptible. The value of $\frac{1}{\xi}P(\varepsilon_p)$ is correct to within 5%.

The curves for the cases where we keep terms up to $O(x^{-1/2})$ and $O(x^{-1})$ have been plotted in Figure 1a, marked "First Approximation" and "Second Approximation" respectively. The integral curve for the second approximation is given in Figure 1b. The Landau curve, marked "$\lambda_w = 1.48$", is given for comparison. The second approximation is definitely better than the first; the agreement is quite good near and to the left of the peak, and the general behavior of the main part of the curve to the right of the peak is correct, although the approximations show the violent oscillations which seem typical of this type of expansion. The curve $1/(\varepsilon-\varepsilon_p)^2$ (and its integral), which represents the single collision law and is asymptotic to the Landau curve for large ε, is also plotted in Figure 1a and 1b. It will be seen that with only the second approximation and the $1/(\varepsilon-\varepsilon_p)^2$ curves as a guide, one could draw a curve which is a pretty good approximation to the correct curve. The third approximation curve $O(x^{-3/2})$ was also computed, but was not plotted in order not to clutter up the figure. It gives no better agreement than the second approximation, and is even a little worse. Further computations with this method will be made using terms up to $O(x^{-1})$.

The value 1.48 will be taken as the limiting value of the parameters b and λ_w for $\mathcal{E}_m' \to \infty$. One could compute these parameters numerically from the curve given by Landau, using 15.23-15.27, but it would hardly be worth the labor, as they are to be used only as convenient parameters in terms of which to plot the curves. It will be seen from Figure 2 that the value $\sigma_w = 1.48\xi$ is about the value required to fit the Landau curve as closely as possible to a Gaussian curve near its peak. (The Gaussian curve is marked "$\lambda_w = 0$".)*

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* Note added in proof:

The author now realizes that the exact values of λ_w and b can be computed readily whenever $\mathcal{E}_p$ is known. Comparing the formula:

$$\widetilde{P}(-i\omega) = e^{x\mathcal{V}_0(s_0)+s_0\mathcal{E}} e^{-(\mathcal{E}-x\mathcal{V}_1(s_0))(i\omega) + \frac{x\mathcal{V}_2(s_0)}{2!}(i\omega)^2 - \frac{x\mathcal{V}_3(s_0)}{3!}(i\omega)^3 + \cdots}$$

from which 15.11 can be derived, with the formula: (See 12.4)

$$\widetilde{P}_1(-i\omega) = e^{-(\mathcal{E}-\bar{\mathcal{E}})(i\omega) + \frac{\sigma_w^2}{2!}(i\omega)^2 - \frac{\lambda_w\sigma_w^3}{3!}(i\omega)^3 + \cdots}$$

from which 14.15 can be derived, we see that σ_w, λ_w, (etc.), as given by 15.10, must agree with the definitions 15.23-15.27, provided only that s is so chosen that $\mathcal{E}_p$ as given by 15.10 is really the abscissa of the peak of the curve $P(\mathcal{E}, x)$.

Since we know that $\mathcal{E}_p$ for the Landau curve is given by 16.9 with $j = 0.37$, we need only set (16.17):

$$\log a = j + C = 0.577 + 0.37 = 0.943$$

$$a = 2.58$$

The true values of λ_w and b are given by 16.17:

$$\lambda_w = a^{1/2} = 1.606$$

$$b = a^{1/2} = 1.606$$

Making use of the curve of j vs $1/c$ given by Figure 3, one could then similarly compute more accurate values for λ_w and b for other values of $1/c$ for which the dotted curve ($\beta^2=0$) differs from the solid curve.

The effect of these corrections would be to change the values of the parameter λ_w associated with the distribution curves for $\lambda_w > 1.2$, and to increase slightly the values of σ_w for these curves and hence to shrink the curves of Figure 2 slightly along the axis of abscissas. The end of the curve F vs λ_w (Fig.2) would be smoothed out, as the curve would end at $F=0.720$, $\lambda_w=1.606$. The dependence of the integral curves of Figure 2 on λ_w would be made more uniform. Figures 3 and 4 would also be affected.

It is unfortunate that this point was not realized until too late to correct the curves which had been prepared for this thesis. Fortunately, however, the distribution curves are still correctly given by Figures 2 - 10. The curves are all interconsistent, and the parameters λ_w, σ_w determined from Figures 3 - 9 are the correct parameters to use in interpreting the distribution curves given by Figure 2. It only happens that these parameters, for $\lambda_w >1.2$, are not quite the true weighted parameters defined by equations 15.23 - 15.27.

It is the intention of the author to prepare at the earliest opportunity a set of distribution curves based on the true values of the weighted parameters. The actual energy loss distributions determined from such revised curves will of course be the same as those determined from Figures 2 - 9 of this thesis.

The values of the parameters when $c<\infty$ are computed first for the case $\beta^2=0$. We compute as functions of c/a the quantities:

$$g = b/a^{1/2} = (1-e^{-c/a})^{1/2}$$

$$h = \lambda/a^{1/2} = \frac{1-(1+c/a)e^{-c/a}}{(1-e^{-c/a})^{3/2}} \qquad 16.18$$

$$k = \zeta/2\lambda^2 = \frac{[1-e^{-c/a}][1-(1+c/a+\tfrac{1}{2}c^2/a^2)e^{-c/a}]}{[1-(1+c/a)e^{-c/a}]^2}$$

The formulas on the right are valid only for $\beta^2 = 0$. In terms of g, h, k, equation 15.15 for a becomes (we are working to $O(x^{-1})$ always):

$$a = \frac{2g/h}{1 - \frac{gh}{2}(k - 5/6)} \qquad 16.19$$

16.19 gives a as a function of c/a, from which we get c as a function of c/a. b, λ, ζ, and j are then determined from 16.17 and 16.18 as functions of c or c/a. The values of g, h, k, a, c, b, λ, and j, computed for $\beta^2 = 0$, are given in Table 3 as functions of c/a. For small values of c, approximate formulas are given at the bottom of the table; the point below which the formulas are accurate to within about 1% is marked by a check in each column of the table. The first term in the approximate formulas for b, λ, and ζ agree with 14.23 and 14.22. The number a varies only from 2.18 to 4 for the entire range $\infty \geq c \geq 0$. The number k, which appears in the relation $\zeta = 2k\lambda^2$ varies only from 1 to 2/3. which suggests that the shape of the curves can be characterized fairly well by giving λ alone, with the other semi-invariants maintained in a fixed relation to λ. This observation we will want to extend later to cover all values of β^2, as well as to cover the case where the collision spectrum is a function of ε.

The parameters b, λ, and j computed in this way are plotted against $1/c$ in Fig. 3; these are the curves marked "$\beta^2 = 0$". The dotted curve in the j graph is the computed curve; the solid curve was obtained by joining it smoothly to the Landau value $j = 0.37$, which is valid for large values of c. As the considerations in the next section will indicate, we expect the true value of j to be constant for c greater than about 4, and to decrease monotonically for smaller values of c. The approximate j rises slightly toward the Landau value 0.37 as the convergence improves with decreasing c, and reaches a maximum 0.32 at 3.5, decreasing monotonically thereafter. The b curve is practically constant for $c > 5$, and then decreases, in agreement with the fact that the width $(\sim b\xi)$ of the curve is at first proportional to ξ, and then increases more slowly, being ultimately proportional to $\xi^{1/2}$. The λ curve is asymptotic to its Landau value 1.48, but begins to decrease appreciably for $c < 12$, since the tail of the curve is reduced, as ϵ'_w/ξ decreases, before the main part of the curve near the peak is affected. An approximate curve computed from 15.13 for the case $\lambda_w = 1$, using terms up to $O(\kappa^{-1})$ is given in Figure 1c. The correct curve, computed by a method to be given in the next section, is also given. The agreement is very good, particularly near and to the left of the peak. A small oscillation in the tail is still evident. The unweighted Edgeworth expansion appears to give a better approximation to the right of the peak.

The number k does not vary greatly when β^2 varies from 0 to 1. If we define a k_u for the unweighted parameters, which are more sensitive to β^2 than the weighted parameters, we have, by 14.22:

$$k_u = \zeta_u/2\lambda_u^2 = \frac{2}{3}\,\frac{(1-\frac{3}{4}\beta^2)(1-\frac{1}{2}\beta^2)}{(1-\frac{2}{3}\beta^2)^2} \qquad 16.20$$

The factor depending on β^2 varies smoothly from 1 to 1.125 as β^2 varies from 0 to 1. Thus for fairly large x, when the unweighted Edgeworth expansion is valid, and the term $O(x^{-1})$ which involves ζ_u is only a small correction, the curves computed for $\beta^2=0$ will do for any value of β^2, provided the correct values of λ and σ are used. This observation is also valid for fairly small values of x, as will be seen by comparing the two curves in Figure 1d, which were computed by the methods of the next section.

Since the value of k occurs in a small correction term in a (Eq.16.19), we are going to assume that the correction factor to be applied to a when $\beta^2 \neq 0$ can be obtained by using the approximate formula:

$$a = 2\,g/h = 2\,b/\lambda \qquad 16.21$$

Using 16.15, 16.16, this gives:

$$a = 2\,b^t/\lambda^t \, \frac{\left(1-\frac{\beta^2}{c} b^t \lambda^t\right)^2}{\left(1-\frac{\beta^2}{c} \frac{b^t \zeta^t}{\lambda^t}\right)}$$

$$= a^t \, \frac{\left(1-\frac{\beta^2}{c} b^t \lambda^t\right)^2}{\left(1-\frac{\beta^2}{c} \frac{b^t \zeta^t}{\lambda^t}\right)} \qquad 16.22$$

We thus get for the correction factor to be applied to a^t:

$$\frac{a}{a^t} = \frac{\left(1-\beta^2 \frac{a}{c} g^t h^t\right)^2}{\left(1-2\beta^2 \frac{a}{c} k^t g^t h^t\right)} \qquad 16.23$$

where g^t, h^t, k^t are the functions of c/a given in 16.18. The correction factor is now a function of c/a and β^2 only, and can be easily computed, since g^t, h^t, k^t were already computed in the previous calculations. a is obtained as a function of c/a and β^2 by multiplying a/a^t by the value a^t of a computed for $\beta^2=0$. c is then immediately determined; and b, λ, and j are computed from 16.15, 16.16, 16.10. The results are given in Table 4, and plotted in Figure 3. Significant deviations from the values in the Landau limit $c \to \infty$, occur at much larger values of c for larger values of β^2. Figure 1d shows that, when the correct curves for $\beta^2=1$ and $\beta^2=0$, computed for values of c corresponding to the same value of λ_w according to Figure 3, are plotted against $\frac{\varepsilon-\varepsilon_p}{\sigma_w}$ with σ_w obtained from Figure 3, they do in fact agree to a sufficiently good approximation for statistical purposes.

17. Other Methods.

Let us suppose that $\chi(E')$ can be written as the sum of two functions:

$$\chi(E') = \chi_I(E') + \chi_{II}(E') \qquad 17.1$$

For convenience, we omit the term X; X may be regarded as included in χ_I or χ_{II}. By 10.10, we have:

$$\tilde{P}(s,x) = e^{x[\tilde{\chi}_I(s)-\tilde{\chi}_I(0)]}\, e^{x[\tilde{\chi}_{II}(s)-\tilde{\chi}_{II}(0)]} \qquad 17.2$$

Using 17.2, 10.10, and the converse of theorem II, Section 10, (22,p.168):

$$P(\varepsilon,x) = \int_0^{\varepsilon} P_I(\varepsilon',x)\, P_{II}(\varepsilon-\varepsilon',x)\, d\varepsilon' \qquad 17.3$$

where $P_I(\varepsilon,x)$, $P_{II}(\varepsilon,x)$ are the energy loss distributions corresponding to the partial collision spectra χ_I and χ_{II}:

$$P_I(\varepsilon,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\varepsilon + x[\tilde{\chi}_I(s)-\tilde{\chi}_I(0)]}\, ds \qquad 17.4$$

$$P_{II}(\mathcal{E},x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\mathcal{E} + x[\tilde{\chi}_{II}(s) - \tilde{\chi}_{II}(0)]}\, ds$$

17.3 is just the formula for the composition of two independent random variables; since they are independent of $\mathcal{E}$, the two partial collision spectra χ_I and χ_{II} may be regarded as causing independent losses of energy. We may thus transform our problem into one in which the particle first traverses a distance x in a material characterized by χ_I, and then a distance x in a material characterized by χ_{II}. $P_I(\mathcal{E},x)$ and $P_{II}(\mathcal{E},x)$ are the distributions of energy losses in the two materials separately, and we get the distribution in total energy loss from 17.3. It should be remarked that the spectra $\chi_I(E')$ and $\chi_{II}(E')$ need not be physically realizable, and may even take on negative values for some values of E'. In the latter case, P_I and P_{II} may take on negative values. (A negative $\chi(E')$ would represent a probability for particles to be removed from the energy loss range $(\mathcal{E}, \mathcal{E}+d\mathcal{E})$ and transferred to the range $(\mathcal{E}-E', \mathcal{E}-E'+d\mathcal{E})$, but the number of particles so transferred would be proportional to the number already at $(\mathcal{E}-E')$,* and not to the number at $\mathcal{E}$, as would be the case if the particles could simply gain energy in a collision. The latter possibility would show up as a non-zero value of $\chi(E')$ for negative values of E'; this kind of a $\chi(E')$ is not likely to be of any use for the present purpose, unless we are dealing with a problem where such collisions really can take place.)

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* See the fundamental equation 9.4.

Let us expand the factor $e^{x\tilde{\chi}(s)}$ in the integrand of 10.10 in a power series, where we suppose that $X=0$:

$$P(\varepsilon,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\varepsilon - x\tilde{\chi}(0)}\left(1+x\tilde{\chi}(s)+\frac{[x\tilde{\chi}(s)]^2}{2!}+\frac{[x\tilde{\chi}(s)]^3}{3!}+\dots\right)ds \qquad 17.5$$

Integrating 17.5 term by term:

$$P(\varepsilon,x) = e^{-x\tilde{\chi}(0)}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\varepsilon}[x\tilde{\chi}(s)]^k ds \qquad 17.6$$

Define the distribution $P_{(n)}(\varepsilon,x)$ by:

$$P_{(n)}(\varepsilon,x) = \frac{1}{n!}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\varepsilon}[x\tilde{\chi}(s)]^n ds \qquad 17.7$$

Using 10.5, and the converse of Theorem II (and III), Section 10, we obtain:

$$\begin{aligned}
P_{(0)}(\varepsilon,x) &= \delta(\varepsilon)\\
P_{(1)}(\varepsilon,x) &= x\chi(\varepsilon)\\
P_{(2)}(\varepsilon,x) &= \frac{1}{2}\int_0^{\varepsilon} x\chi(\varepsilon')x\chi(\varepsilon-\varepsilon')d\varepsilon'\\
P_{(3)}(\varepsilon,x) &= \frac{1}{6}\int_0^{\varepsilon} d\varepsilon' x\chi(\varepsilon')\int_0^{\varepsilon-\varepsilon'} d\varepsilon'' x\chi(\varepsilon'')x\chi(\varepsilon-\varepsilon'-\varepsilon''),\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
P_{(k+1)}(\varepsilon,x) &= \frac{1}{(k+1)}\int_0^{\varepsilon} P_{(k)}(\varepsilon',x)x\chi(\varepsilon-\varepsilon')d\varepsilon'
\end{aligned} \qquad 17.8$$

Thus $P_{(n)}(\varepsilon,x)$ is the probability that the particle loses a total energy ε in n collisions in going a distance x. 17.6 becomes:

$$P(\varepsilon,x) = \sum_{k=0}^{\infty} e^{-x\tilde{\chi}(0)}P_{(k)}(\varepsilon,x) \qquad 17.9$$

The interpretation of 17.9 is obvious. The kth term gives the contribution to $P(\varepsilon,x)$ due to those particles which suffer just k collisions. The factor $e^{-x\tilde{\chi}(0)}$ represents the decay in the number of particles in any given group due to removal of particles by collisions, which occur at a rate of $\tilde{\chi}(0)$ collisions per unit distance. Since $P_{(k)}(\epsilon, x)$ is proportional to x^k, each term $e^{-x\tilde{\chi}(0)} P_{(k)}(\epsilon, x)$ builds up to a maximum at $x = {}^{k}/\tilde{\chi}(0)$ and then decays exponentially. If $\tilde{\chi}(0)$ is large (as in the case of 8.1), 17.9 is of little value except for extremely small values of x. When $\tilde{\chi}(0)$ is small, the first few terms of 17.9 may give a good approximation to $P(\epsilon, x)$. Here, as in 17.3, the mathematical derivation of 17.9 can be carried out in cases when $\chi(E')$ is not a physically realizable collision spectrum.

If we can split χ into two functions χ_I and χ_{II} such that the solution for χ_I is known, and $\tilde{\chi}_{II}(0)$ is small, we can use 17.9 and 17.3 to find the solution for the spectrum χ. This method was used to compute the curves of Figure 1d, taking χ_I as the Landau collision spectrum 11.1. For $\beta^2 = 0$, the computation of $P(\epsilon, x)$, using the Landau curve for $P_I(\epsilon, x)$, requires only terms up to $k = 2$ or 3 in 17.9 for $\lambda_w \geq 1$. For $\beta^2 = 1$, the computation becomes rather laborious for $\lambda_w < 1.3$. This method was also used by Williams (2) to obtain his result for the Landau curve by taking χ_I as 8.1 with $E'_m = E'_T$ and $\beta^2 = 0$, where E'_T is chosen small enough to make the function $P_I(\epsilon, x)$ approximately Gaussian. (See Section 2.) χ_{II} was then taken as the remainder of the spectrum $E'_T \leq E' < \infty$, and $P_{II}(\epsilon, x)$ computed from 17.9. A better method would probably be to use either the weighted or unweighted Edgeworth expansion (or both) to get P_I, choosing E'_T to make $\lambda \leq 1$. For a given E'_T, this gives a much better approximation to P_I than a Gaussian function, or, alternatively, it allows the use of a much larger E'_T for the same degree of accuracy in P_I, with consequent saving of labor in computing P_{II}.

Let us consider a large number N of particles traversing a distance x in a material characterized by 8.1 with $\beta^2 = 0$. We suppose that $C = \frac{E'_m}{\xi}$ is moderately large, but not infinite, The resulting distribution $P_I(\mathcal{E},x)$ will be similar to one of those of Figure 2, with $\lambda_w < 1.48$. We now allow the particles to traverse a distance x in a second material characterized by:

$$\nu_{II}(E') = \begin{cases} 0 & (E' < E'_m) \\ \frac{2\mu_e}{\beta^2}\frac{1}{E'^2} & (E' > E'_m) \end{cases} \qquad 17.10$$

The number of particles which suffer no additional collisions in the second material is $N e^{-x\bar{\nu}_{II}(0)} = N e^{-1/c}$; the distribution of these particles is just $e^{-1/c} N P_I(\mathcal{E},x)$. The $N(1-e^{-1/c})$ particles which suffer collisions in the second material are all at least an amount E'_m farther down the energy scale than they were after leaving material I, hence their distribution $NR_{II}(\mathcal{E},x)$ * begins at $\mathcal{E}_{II} = \mathcal{E}_I + E'_m$, where $\mathcal{E}_I$ is the smallest energy loss for which $P_I(\mathcal{E},x)$ has an appreciable value.

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* We depart slightly from the previous notation in that $\mathcal{E}$ in $R_{II}(\mathcal{E},x)$ is the sum of the energy lost in materials I and II, and $R_{II}(\mathcal{E},x)$ includes only particles which lose some energy in II. In $P_{II}(\mathcal{E},x)$ as previously used, (Eq.17.4), $\mathcal{E}$ was the energy lost in material II, and $P_{II}(\mathcal{E},x)$ included all particles.

The complete final distribution is $NP_L(\mathcal{E},x) = Ne^{-1/c}P_I(\mathcal{E},x) + NR_{II}(\mathcal{E},x)$. $P_L(\mathcal{E},x)$ is the Landau distribution function. $P_L(\mathcal{E},x)$ and $e^{-1/c}P_I(\mathcal{E},x)$ will coincide for $\mathcal{E}$ between $\mathcal{E}_I$ and $\mathcal{E}_I + E'_m$; and for $\mathcal{E} > \mathcal{E}_I + E'_m$, $P_L(\mathcal{E},x)$ will have a larger tail. Hence the curves $e^{-1/c}P(\mathcal{E},x)$ for the collision spectrum 8.1 with $\beta^2 = 0$, if plotted against $\Delta_L = \frac{\mathcal{E} - \mathcal{E}_{p(L)}}{\xi}$, will coincide with the Landau curve $P_L(\mathcal{E},x)$ from $\Delta_L \sim -4$ (see Fig. 1a or 1c) to $\Delta_L \sim (-4+c)$, and will fall below $P_L(\mathcal{E},x)$ beyond this point. For $c \gtrsim 4$, the curves will coincide up to or beyond the peak $\Delta_L = 0$; this confirms the statement in the last section that we should expect j to be constant for $c \gtrsim 4$, if $\beta^2 = 0$.

The same remarks will apply if we plot against $\Delta_W = \frac{\mathcal{E} - \mathcal{E}_p}{b\xi}$, for all values of c for which $b \doteq 1.48$. Fortunately, b is practically constant for values of c down to about 4 ($\beta^2 = 0$), i.e. $\lambda_W \gtrsim 1$, so that this behavior of the curves is preserved when they are multiplied by an appropriate factor and plotted against Δ_W as in Figure 2. For $c < 4$, the method of plotting in Figure 2 is such as to fit all the curves as nearly as possible near the peak. (See the next section.) For a given value of c, the curves for larger values of β^2 will peel away from the Landau curve sooner than for $\beta^2 = 0$, since the β^2 term in 8.1 reduces the collision spectrum $\gamma(E')$ for values of E' less than E'_m. Correspondingly, the values of λ_W (Fig. 3), for a given value of c, decrease as β^2 increases.

For very large values of c, the curve $P(\mathcal{E},x)$ for $\beta^2 = 0$ may be obtained approximately from the Landau limiting curve

$P_L(\mathcal{E},x)$ by multiplying by $e^{1/c}$ (usually a negligible correction), and cutting the tail of the curve off at $\mathcal{E} = \mathcal{E}_p + E'_m$. This corresponds to assuming that any particle at a point in the tail of the Landau curve has arrived there by a single collision in which it lost an energy $\mathcal{E}-\mathcal{E}_p$; if $\mathcal{E}-\mathcal{E}_p > E'_m$, then with the correct spectrum, the particle would not have suffered such a collision. A somewhat better approximation would be to set:

$$P(\mathcal{E},x) = e^{1/c} P_L(\mathcal{E},x) \qquad (\mathcal{E} < \mathcal{E}_p - 4\xi + E'_m)$$

$$(c >> 1) \qquad 17.11$$

$$P(\mathcal{E},x) = P(\mathcal{E}_p - 4\xi + E'_m, x)\left[1 - \int_0^{\mathcal{E}-E'_m} P(\mathcal{E},x)\,d\mathcal{E}\right] \qquad (\mathcal{E} > \mathcal{E}_p - 4\xi + E'_m)$$

This cuts the tail off gradually, beginning at about $\mathcal{E} = \mathcal{E}_p - 4\xi + E'_m$. It corresponds roughly to assuming that any particle near the end of the tail in the correct curve $P(\mathcal{E},x)$ has arrived there by losing about E'_m in a single collision; as $\mathcal{E}$ becomes greater than $\mathcal{E} - 4\xi + E'_m$, we apply a factor to correct for the fact that particles in the early part of the distribution can no longer reach $\mathcal{E}$ by a single collision. 17.11 gives a good approximation when $\lambda_w > 1.4$. When $\beta^2 \neq 0$, the approximate curve for large c, or $\lambda_w > 1.4$, may be obtained by multiplying $P_L(\mathcal{E},x)$ by $\left[1 + \int_{10\xi}^{\infty} [\chi_L(E') - \chi(E')]\,dE'\right]$, (where $\chi_L(E')$ is the spectrum 11.1 and $\chi(E')$ is the correct spectrum 8.1) and replacing the tail of the Landau curve by the single collision distribution $\chi(\mathcal{E}-\mathcal{E}_p)$ for large values of $\mathcal{E}$ (say $\mathcal{E}-\mathcal{E}_p > 10\xi$). If $\beta^2 < 1$ there will be an abrupt cut-off in the tail at $\mathcal{E}_p + E'_m$, which could be avoided by a device similar to that used in 17.11.

18. Construction of the Distribution Curves.

The method of obtaining the weighted parameters $\mathcal{E}_p$, σ_w, λ_w as functions of C and β^2 has been described in Section 16. It was shown at the end of Section 16 that to a good approximation all the distribution curves which result from a spectrum of the form 8.1 can be characterized by these three parameters alone. Variations in the first two parameters are transformed away by choosing $\Delta_w = \frac{\mathcal{E}-\mathcal{E}_p}{\sigma_w}$ as the variable characterizing the energy loss, and plotting the functions:

$$P_{\lambda_w}(\Delta_w) = \sigma_w P(\mathcal{E},x) \qquad 18.1$$

The functions $P_{\lambda_w}(\Delta_w)$ are normalized in the scale of the variable Δ_w, and are a one parameter family of curves characterized by the single parameter λ_w. The relations between the distribution curves for different values of λ_w are more clearly brought out, and interpolation between the curves for intermediate values of λ_w is greatly facilitated, by plotting $\frac{1}{F}P_{\lambda_w}(\Delta_w)$, where F is a factor chosen so that the curves all coincide at their peaks ($\Delta_w = 0$). The curves then agree very closely in the neighborhood of $\Delta_w = 0$ and fan out in a regular way as we go away from the peak. This behavior can be seen in Figure 2, where the family of curves $\frac{1}{F}P_{\lambda_w}(\Delta_w)$ is plotted. The curves for $\lambda_w \geq 1$ coincide to the left of the peak as explained in the preceding section.

The curve $\lambda_w = 0$ is simply a normalized Gaussian curve with unit standard deviation. For a Gaussian curve, $\sigma_u = \sigma_w$, and $\lambda_w = \lambda_a = 0$. The curve $\lambda_w = 0.5$ was obtained to the left of the peak from a weighted Edgeworth expansion, and to the right from an ordinary Edgeworth expansion, with $\beta^2 = 0$, and choosing $c = 0.95$, which corresponds to $\lambda_w = 0.5$. The curve $\lambda_w = 0.8$ is plotted only to the right of the peak, and was obtained from an unweighted Edgeworth expansion, with $\beta^2 = 0$ and $c = 2.4$. The curve $\lambda_w = 1.48$ was taken from Landau's paper (3) with appropriate change of scale. The curves $\lambda_w = 1.0$, 1.2, 1.4 were obtained by the method described in Section 17. The curve $\lambda_w = 1.0$ is essentially the same as would be obtained from an Edgeworth expansion, as shown by Figure 1c. The curves $\lambda_w = 1.0$ and 1.2 were calculated for $\beta^2 = 0$. The curve $\lambda_w = 1.4$ was computed for both $\beta^2 = 0$ and $\beta^2 = 1$ (Fig. 1d); the curve plotted in Figure 2 is intermediate between the two.

The factor F required to make the peak values coincide with the peak ordinate 0.399 of the Gaussian curve was determined for $\lambda_w < 1$ by computing the value of $\sigma_w P(\mathcal{E}_p)$ from the Edgeworth series and setting:

$$F = \frac{\sigma_w P(\mathcal{E}_p)}{0.399} \qquad 18.2$$

The factor F for the Landau curve is:

$$\begin{aligned} F_L &= \frac{\sigma_w P_L(\mathcal{E}_p)}{0.399} \\ &= \frac{b\,\phi(0)}{0.399} \\ &= \frac{1.478 \times 0.179}{0.399} \\ &= 0.663 \end{aligned} \qquad 18.3$$

where ϕ is the function given by Landau (Eq.11.8). For $\lambda_w > 1$, use was made of the result of the previous section that the peaks will coincide if we set:

$$F_{\lambda_w} = e^{1/c} F_L \qquad 18.4$$

where c is the value of E'_w/ξ corresponding to λ_w for $\beta^2 = 0$. The curves for F as a function of λ_w computed from 18.2 and 18.4 join smoothly at $\lambda_w \sim 1$. This curve is plotted in Figure 2.

The integral curves in Figure 2 were obtained by integrating the series used for $\lambda_w < 1$, and by integrating numerically the computed curves for $\lambda_w \geq 1$. The normalized integrals $\int_{\mathcal{E}}^{\infty} P(\mathcal{E},x)\,d\mathcal{E}$ were plotted, as it was felt that the curves would be more convenient to use in this form. If we plot instead $\frac{1}{F}\int_0^{\mathcal{E}} P(\mathcal{E},x)\,d\mathcal{E}$, then the integral curves exhibit the same regularity as the differential curves; they all very nearly coincide for $\Delta_w < 0$, and for positive Δ_w, the curves peel off one by one and approach their respective horizontal asymptotes corresponding to the respective values of $1/F$. The differential curves $\frac{\sigma_w}{F} P(\mathcal{E},x)$ coincide, for $\lambda_w \geq$ 1.45, with the Landau curve throughout the range of Δ_w plotted in Figure 2. Hence the integral curve for $\lambda_w \geq$ 1.45 may be obtained from:

$$\int_0^{\Delta_w} P_{\lambda_w}(\Delta_w)\,d\Delta_w = \frac{F_{\lambda_w}}{F_L}\int_0^{\Delta_w} P_{\lambda_w=1.48}(\Delta_w)\,d\Delta_w \qquad \begin{pmatrix}\Delta_w \leq 10, \\ \lambda_w \geq 1.45\end{pmatrix} \qquad 18.5$$

where F_{λ_w} is taken from Figure 2 or, better, computed from (Sec.17):

$$F_{\lambda_w} = F_L \left[1 + x\int_{10\xi}^{\infty}[\chi_L(E') - \chi(E')]\,dE'\right]$$

$$= F_L \left(1 + 1/c + \beta^2/c \log \frac{c}{10}\right) \qquad 18.6$$

Had we used the unweighted parameters $\bar{\mathcal{E}}$, σ_u, λ_u, and plotted the functions:

$$P_{\lambda_u}(\Delta_u) = \sigma_u P(\mathcal{E},x) \qquad \left(\Delta_u = \frac{\mathcal{E}-\bar{\mathcal{E}}}{\sigma_u}\right) \tag{18.7}$$

we would have obtained a set of curves whose peaks get narrower and taller as λ_u increases. In the Landau limit, $\bar{\mathcal{E}}$, σ_u, λ_u are all infinite, the peak is infinitely narrow and infinitely high, and its abscissa is $\Delta_u = 0$. For $\lambda_u > 1$, the curves would be different for different values of β^2. The weighted parameters give a way of plotting the whole range of curves on a scale convenient for all.

The differential and integral curves plotted in Figure 2 are tabulated numerically in Table 5.

19. Relations between Weighted and Unweighted Parameters.

The weighted parameters $\mathcal{E}_p$, σ_w, λ_w are in general more difficult to compute than the unweighted parameters $\bar{\mathcal{E}}$, σ_u, λ_u, even in the case when the collision spectrum is independent of $\mathcal{E}$. When $\chi(E')$ depends on $\mathcal{E}$, the fundamental equation can not be solved, as in Section 10, by a Laplace transformation, and it becomes completely impractical to determine directly the weighted parameters. The unweighted parameters can, however, still be determined, as will be shown in the next chapter. We must therefore be able to transform from unweighted to weighted parameters in order to use the curves of Figure 2.

Given λ_w and β^2, we can determine c from Figure 3, and λ_u and σ_u are then given by 14.24 and 14.23. Values of λ_u vs λ_w obtained in this way are plotted in Figure 4. As $\lambda_w \to 1.48$, $\lambda_u \to \infty$. For $\lambda_w \leq 1$, we can take $\lambda_u \doteq \lambda_w$ as a fairly good approximation for all values of β^2. The curves are asymptotic to $\lambda_w = 1.48$ for all values of β^2, but the approach to this asymptote is slower for larger values of β^2. The ratio σ_w/σ_u turns out to be nearly independent of β^2. This ratio is plotted in Figure 4. The points computed for $\beta^2 = 0$ and $\beta^2 = 1$ are indicated by the dots in Figure 4, and the curve drawn through them is a sufficiently good approximation for all values of β^2. The ratio $\sigma_w/\sigma_u \to 0$ as $\lambda_w \to 1.48$.

From 16.9 we obtain:

$$\begin{aligned} \overline{\Delta}_w &= \frac{\overline{\varepsilon} - \varepsilon_p}{\sigma_w} \\ &= \frac{\overline{\varepsilon} - \varepsilon_p}{b\overline{\varepsilon}} \\ &= \frac{1}{b}\left(\log c - j - \beta^2\right) \qquad 19.1 \end{aligned}$$

From this equation and Figure 3, we can determine $\overline{\Delta}_w$, which is plotted against λ_w in Figure 4. The result is essentially independent of β^2. The conversion from weighted to unweighted parameters and vice versa is readily carried out by means of the curves given in Figure 4. The numerical values from which Figure 4 was plotted are given in Table 6.

It should be pointed out that $\left(\sigma_u/\sigma_w\right)^2$ is just the mean square fluctuation of the variable Δ_w, measured from its mean value $\overline{\Delta_w}$:

$$\sigma_u^2 = \int_0^\infty (\epsilon - \bar{\epsilon})^2 P(\epsilon, x)\, d\epsilon$$

$$= \sigma_w^2 \int_{-\infty}^{\infty} (\Delta_w - \overline{\Delta_w})^2 P_{\lambda_w}(\Delta_w)\, d\Delta_w \qquad 19.2'$$

where $P_{\lambda_w}(\Delta_w)$, defined by 18.1, is given by the curves of Figure 2. Hence $(\sigma_u/\sigma_w)^2$ and $\overline{\Delta_w}$ are determined when $P_{\lambda_w}(\Delta_w)$ is given. To the extent that the distributions for all values of β^2 and C can be represented by a one parameter family of curves $P_{\lambda_w}(\Delta_w)$, the quantities σ_w/σ_u and $\overline{\Delta_w}$ must depend only on λ_w. The fact that they are very nearly independent of β^2 is an additional confirmation of the possibility of characterizing the shape of the curves approximately by the single parameter λ_w. The parameter λ_u is also determined once the shape of the curve is given:

$$\lambda_u = \left(\frac{\sigma_w}{\sigma_u}\right)^3 \int_{-\infty}^{\infty} (\Delta_w - \overline{\Delta_w})^3 P_{\lambda_w}(\Delta_w)\, d\Delta_w \qquad 19.3$$

The parameter λ_u is, however, much more sensitive to variations in the tails of the curves than the weighted parameter λ_w. The effect of the β^2 term in 8.1 is to make the tail of the collision spectrum thinner ($\chi(E')$ smaller) for large values of E'. This results in a similar effect on the energy loss distribution; $P(\epsilon, x)$ has a thinner tail. Both λ_u and λ_w are in general larger for a long thin tail than for a short thick tail, but the effect on the weighted skewness is smaller, since in computing λ_w, less weight is given to the end of the tail. By choosing curves with the same value of λ_w, we match their shapes as nearly as possible over the main part of the curve (see Fig. 1d), but the curve with the longer, thinner tail ($\beta^2 = 1$) will have a larger λ_u than that with the shorter, thicker tail ($\beta^2 = 0$). These qualitative considerations are expressed quantitatively in Figure 4, where the λ_u vs λ_w curves for different values of β^2 are plotted.

In concluding this chapter, it may be remarked that the basic assumption on which the results so far depend, that $\chi(E')$ is independent of $\mathcal{E}$, may be expected to be valid at least for thicknesses x for which the mean energy loss $\bar{\mathcal{E}}$ is a small fraction of the initial kinetic energy, say $\bar{\mathcal{E}} < 0.1 E_k^0$. The variation of $\chi(E')$ with $\mathcal{E}$ at high energies ($\beta^2 \sim 1$) consists almost entirely in the variation of E'_m (see Eq. 8.1). At lower energies ($\beta^2 < 1$), the variation of the factor $1/\beta^2$ in $\chi(E')$ must also be taken into account. There is also a small variation in the value of $X = \int_0^\eta \chi(E')\,dE'$ due to the variation of $\log 1/\epsilon$. The limits of validity of the results of the present chapter can be more precisely determined from the results of the next chapter.

CHAPTER IV

SOLUTION FOR A COLLISION SPECTRUM DEPENDENT ON THE ENERGY

20. Equations of Motion for the Moments in the General Case.

Let us consider the fundamental equation in the form 9.8. We expect that the fluctuation $(\overline{(\Delta \mathcal{E})^2})^{1/2}$ in the energy loss will be a small fraction of the mean kinetic energy $\bar{E}_k$, over most of their range for particles with a given initial energy E_k°. Therefore the collision spectrum $\chi(E';\mathcal{E},x)$ will, for nearly all particles, be practically the spectrum evaluated at the mean energy $\chi(E';\bar{\mathcal{E}}(x),x)$, where $\bar{\mathcal{E}}$, as indicated, is a function of the path length x. It is therefore suggested that we seek an approximate solution to 9.8 by substituting $\chi(E';\bar{\mathcal{E}}(x),x)$ for $\chi(E';\mathcal{E},x)$. The equation then reduces to one of the form 10.2, and its solution can be carried out formally by a Laplace transformation just as in Chapter II. We define the moments of the collision spectrum as in 12.1:

$$\chi_1(\mathcal{E},x) = I(\mathcal{E},x) + \int_0^\infty E'\chi(E';\mathcal{E},x)\,dE'$$

$$\chi_n(\mathcal{E},x) = \int_0^\infty E'^n\chi(E';\mathcal{E},x)\,dE' \qquad (n>1) \tag{20.1}$$

In analogy with 12.16, we would have for $\bar{\mathcal{E}}$:

$$\bar{\mathcal{E}}(x) = \int_0^x \chi_1(\bar{\mathcal{E}}(x),x)\,dx \tag{20.2}$$

This is an integral equation to be solved for $\bar{\mathcal{E}}(x)$. In analogy with 12.17, we would then compute the higher moments:

$$\overline{(\Delta \mathcal{E})^2} = \int_0^x \chi_2(\bar{\mathcal{E}}(x),x)\,dx$$

$$\overline{(\Delta \mathcal{E})^3} = \int_0^x \chi_3(\bar{\mathcal{E}}(x),x)\,dx \qquad 20.3$$

$$\overline{(\Delta \mathcal{E})^4} = \int_0^x \chi_4(\bar{\mathcal{E}}(x),x)\,dx + 3\left(\overline{(\Delta \mathcal{E})^2}\right)^2$$

etc.

The formula 20.3 for $\overline{(\Delta \mathcal{E})^2}$ is quoted by Bethe and Livingston (10, p.282), and was given essentially by Bohr in 1915 (1). It might be expected to be valid so long as $\left(\overline{(\Delta \mathcal{E})^2}\right)^{1/2}$ remains sufficiently small so that $\chi(\mathcal{E}',\mathcal{E},x)$ differs negligibly from $\chi(\mathcal{E}',\bar{\mathcal{E}},x)$ whenever $|\mathcal{E}-\bar{\mathcal{E}}| < \alpha\left(\overline{(\Delta \mathcal{E})^2}\right)^{1/2}$, where α is chosen so that only a small percentage of particles suffer an energy loss $\mathcal{E}$ lying outside this limit, say $\alpha = 3$ for less than 1% outside. We may formulate this condition schematically as:

$$\left|\frac{1}{\chi}\frac{\partial \chi}{\partial \mathcal{E}}\right| \alpha \left(\overline{(\Delta \mathcal{E})^2}\right)^{1/2} << 1 \qquad 20.4$$

Unfortunately, 20.4 is not sufficient to guarantee the validity of 20.3, no matter what value of α is chosen, as we shall see. Only the condition:

$$\left|\frac{1}{\chi}\frac{\partial \chi}{\partial \mathcal{E}}\right| \bar{\mathcal{E}} << 1 \qquad 20.5$$

can guarantee the validity of 20.3, so that the formulas 20.3 are little better than the formulas of Chapter II, where the variation of χ with ε was neglected entirely.

Let us multiply the fundamental equation 9.8 by a function $f(\varepsilon,x)$, and integrate over ε from 0 to ∞:

$$\int_0^\infty f(\varepsilon,x)\frac{\partial}{\partial x}P(\varepsilon,x)\,d\varepsilon = \int_0^\infty d\varepsilon\int_0^\varepsilon d\varepsilon'\left[f(\varepsilon,x)P(\varepsilon',x)\chi(\varepsilon-\varepsilon';\varepsilon',x)\right]$$

$$-\int_0^\infty d\varepsilon\int_0^\infty d\varepsilon'\left[f(\varepsilon,x)P(\varepsilon,x)\chi(\varepsilon';\varepsilon,x)\right]$$

$$-\int_0^\infty d\varepsilon\, f(\varepsilon,x)\frac{\partial}{\partial\varepsilon}\left[P(\varepsilon,x)I(\varepsilon,x)\right] \qquad 20.6$$

We integrate the last term by parts, and recognize that $P(\varepsilon,x)$ vanishes at the limits.* In the first integral, we make the substitution:

$\varepsilon-\varepsilon'$ is replaced by ε'

ε' is replaced by ε 20.7

$\int_0^\infty d\varepsilon\int_0^\varepsilon d\varepsilon'$ is replaced by $\int_0^\infty d\varepsilon\int_0^\infty d\varepsilon'$

The first and second integrals can then be combined. We define:

$$\bar{f}(x) = \int_0^\infty f(\varepsilon,x)P(\varepsilon,x)\,d\varepsilon$$

$$\overline{\frac{\partial f}{\partial x}} = \int_0^\infty \frac{\partial f}{\partial x}P(\varepsilon,x)\,d\varepsilon \qquad 20.8$$

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* We are integrating over the entire distribution, hence down to 0^- if $P(\varepsilon,x)$ has a singularity at $\varepsilon = 0$.

With these changes in notation, 20.6 becomes:

$$\frac{d\bar{f}}{dx} = \int_0^\infty d\mathcal{E}\int_0^\infty dE'\left\{P(\mathcal{E},x)\chi(E';\mathcal{E},x)\left[f(\mathcal{E}+E';x)-f(\mathcal{E},x)\right]\right\}$$

$$+\int_0^\infty P(\mathcal{E},x)\mathcal{X}(\mathcal{E},x)\frac{\partial f}{\partial \mathcal{E}}d\mathcal{E} + \overline{\frac{\partial f}{\partial x}} \qquad 20.9$$

Taking $f(\mathcal{E},x)=\mathcal{E}$, 20.9 becomes:

$$\frac{d\bar{\mathcal{E}}}{dx} = \int_0^\infty \chi_1(\mathcal{E},x)P(\mathcal{E},x)\,d\mathcal{E} \qquad 20.10$$

The interpretation of this equation is obvious. $\frac{d\bar{\mathcal{E}}}{dx}$ is simply the first moment of the collision spectrum averaged over the distribution. We now take:

$$f(\mathcal{E},x) = (\Delta\mathcal{E})^n = (\mathcal{E}-\bar{\mathcal{E}})^n$$

$$f(\mathcal{E}+E';x) = (\Delta\mathcal{E})^n + nE'(\Delta\mathcal{E})^{n-1}+\cdots+\binom{n}{k}E'^k(\Delta\mathcal{E})^{n-k}+\cdots+E'^n$$

$$\frac{\partial f}{\partial\mathcal{E}} = n(\Delta\mathcal{E})^{n-1} \qquad 20.11$$

$$\frac{\partial f}{\partial x} = -n(\Delta\mathcal{E})^{n-1}\frac{d\bar{\mathcal{E}}}{dx}$$

With 20.11, 20.9 becomes:

$$\frac{d\overline{(\Delta\mathcal{E})^n}}{dx} = \int_0^\infty\left\{n\chi_1(\mathcal{E},x)(\Delta\mathcal{E})^{n-1}+\cdots+\binom{n}{k}\chi_k(\mathcal{E},x)(\Delta\mathcal{E})^{n-k}+\cdots+\chi_n(\mathcal{E},x)\right\}P(\mathcal{E},x)\,d\mathcal{E}$$

$$-n\,\overline{(\Delta\mathcal{E})^{n-1}}\,\frac{d\bar{\mathcal{E}}}{dx} \qquad 20.12$$

If the collision spectrum were independent of $\mathcal{E}$, 20.10 and 20.12 would reduce to:

$$\frac{d\bar{\mathcal{E}}}{dx} = \chi_1$$

$$\frac{d\overline{(\Delta\mathcal{E})^n}}{dx} = \chi_n + \binom{n}{n-2}\chi_{n-2}\overline{(\Delta\mathcal{E})^2} + \cdots + \binom{n}{k}\chi_k\overline{(\Delta\mathcal{E})^{n-k}} + \cdots + \binom{n}{2}\chi_2\overline{(\Delta\mathcal{E})^{n-2}} \qquad 20.13$$

The solution of 20.13 is given by 20.2 and 20.3.

Let us now proceed a little further with the ideas of the first paragraph, by assuming that the moments of the collision spectrum can be expanded in a power series about $\mathcal{E} = \bar{\mathcal{E}}$:

$$\chi_n(\mathcal{E},x) = \sum_k \frac{1}{k!}\frac{\partial^k\chi_n}{\partial\mathcal{E}^k}(\Delta\mathcal{E})^k \qquad 20.14$$

where the derivative is to be evaluated at $\mathcal{E} = \bar{\mathcal{E}}$. Substituting 20.14 in 20.10 and 20.12:

$$\frac{d\bar{\mathcal{E}}}{dx} = \chi_1 + (0) + \left[\frac{\overline{(\Delta\mathcal{E})^2}}{2}\frac{\partial^2\chi_1}{\partial\mathcal{E}^2}\right] + \left[\frac{\overline{(\Delta\mathcal{E})^3}}{6}\frac{\partial^3\chi_1}{\partial\mathcal{E}^3}\right] + \cdots \qquad 20.15$$

$$\frac{d\overline{(\Delta\mathcal{E})^2}}{dx} = \chi_2 + \left(2\,\overline{(\Delta\mathcal{E})^2}\frac{\partial\chi_1}{\partial\mathcal{E}}\right) + \left[\overline{(\Delta\mathcal{E})^3}\frac{\partial^2\chi_1}{\partial\mathcal{E}^2} + \frac{\overline{(\Delta\mathcal{E})^2}}{2}\frac{\partial^2\chi_2}{\partial\mathcal{E}^2}\right]$$

$$+\left[\frac{2}{3}\overline{(\Delta\mathcal{E})^4}\frac{\partial^3\chi_1}{\partial\mathcal{E}^3} + \frac{\overline{(\Delta\mathcal{E})^3}}{3}\frac{\partial^3\chi_2}{\partial\mathcal{E}^3}\right] + \cdots \qquad 20.16$$

$$\frac{d\overline{(\Delta\mathcal{E})^3}}{dx} = \chi_3 + \left(3\,\overline{(\Delta\mathcal{E})^3}\frac{\partial\chi_1}{\partial\mathcal{E}} + 3\,\overline{(\Delta\mathcal{E})^2}\frac{\partial\chi_2}{\partial\mathcal{E}}\right) + \left[\frac{3}{2}\left(\overline{(\Delta\mathcal{E})^4} - \overline{(\Delta\mathcal{E})^2}^{\,2}\right)\frac{\partial^2\chi_1}{\partial\mathcal{E}^2} + \frac{3}{2}\overline{(\Delta\mathcal{E})^3}\frac{\partial^2\chi_2}{\partial\mathcal{E}^2}\right.$$

$$\left.+\frac{1}{2}\overline{(\Delta\mathcal{E})^2}\frac{\partial^2\chi_3}{\partial\mathcal{E}^2}\right] + \left[\frac{1}{2}\left(\overline{(\Delta\mathcal{E})^5} - \overline{(\Delta\mathcal{E})^3}\,\overline{(\Delta\mathcal{E})^2}\right)\frac{\partial^3\chi_1}{\partial\mathcal{E}^3} + \frac{1}{2}\overline{(\Delta\mathcal{E})^4}\frac{\partial^3\chi_2}{\partial\mathcal{E}^3}\right.$$

$$\left.+\frac{1}{6}\overline{(\Delta\mathcal{E})^3}\frac{\partial^3\chi_3}{\partial\mathcal{E}^3}\right] + \cdots \qquad 20.17$$

$$\frac{d\,\overline{(\Delta\varepsilon)^n}}{dx} = \chi_n + \binom{n}{n-2}\chi_{n-2}\overline{(\Delta\varepsilon)^2} + \cdots + \binom{n}{k}\chi_k\overline{(\Delta\varepsilon)^{n-k}} + \cdots + \binom{n}{2}\chi_2\overline{(\Delta\varepsilon)^{n-2}}$$

$$+\left(n\overline{(\Delta\varepsilon)^n}\frac{\partial\chi_1}{\partial\varepsilon} + \cdots + \binom{n}{k}\overline{(\Delta\varepsilon)^{n-k+1}}\frac{\partial\chi_k}{\partial\varepsilon} + \cdots + \overline{(\Delta\varepsilon)^2}\frac{\partial\chi_{n-1}}{\partial\varepsilon}\right)$$

$$+\left[\frac{n}{2}\left(\overline{(\Delta\varepsilon)^{n+1}} - \overline{(\Delta\varepsilon)^{n-1}}\,\overline{(\Delta\varepsilon)^2}\right)\frac{\partial^2\chi_1}{\partial\varepsilon^2} + \cdots + \frac{1}{2}\binom{n}{k}\overline{(\Delta\varepsilon)^{n-k+2}}\frac{\partial^2\chi_k}{\partial\varepsilon^2} + \cdots\right.$$

$$\left. + \frac{1}{2}\overline{(\Delta\varepsilon)^2}\frac{\partial^2\chi_n}{\partial\varepsilon^2}\right] + \cdots \qquad - 20.18$$

The terms have been arranged in brackets according to the order of derivatives of the moments involved.

The unbracketed terms in 20.15 - 20.18 are the terms which appear also in 20.13; they are independent of the rate of variation of $\chi(E')$ with ε; (they are of course evaluated always at $\varepsilon = \bar{\varepsilon}$). The series of terms in the derivatives of χ_n, χ_{n-2}, χ_{n-3}, . . . , χ_2, which appear in brackets in 20.18 can be identified as representing the average of the unbracketed terms over the distribution $P(\varepsilon,x)$. In addition, there are terms in the derivatives of χ_{n-1} and χ_1. The latter are particularly interesting inasmuch as they would be present even if the term I alone were present in the spectrum. Consider the equation 20.16, keeping only the term in $\frac{\partial\chi_1}{\partial\varepsilon}$:

$$\frac{d\,\overline{(\Delta\varepsilon)^2}}{dx} = 2\,\overline{(\Delta\varepsilon)^2}\frac{\partial\chi_1}{\partial\varepsilon} \qquad 20.19$$

If we keep only the term in χ_1 in 20.15, 20.19 can be written:

$$\frac{1}{2}\frac{d\,\overline{(\Delta\varepsilon)^2}}{\overline{(\Delta\varepsilon)^2}} = \frac{d\sigma_u}{\sigma_u} = \frac{1}{\chi_1}\frac{\partial\chi_1}{\partial\varepsilon}\,d\bar{\varepsilon} \qquad 20.20$$

Integrating 20.20:

$$\frac{\sigma_u}{\sigma_u^0} = \frac{\chi_1(\bar{\mathcal{E}})}{\chi_1(\bar{\mathcal{E}}^0)} \qquad 20.21$$

Thus even when we neglect the terms which produce random variations in the amount of energy lost, there will be a fractional change in σ_u equal to the fractional change in χ_1, and independent of the magnitude of σ_u. The fractional error in σ_u due to neglecting the term in $\frac{\partial \chi_1}{\partial \mathcal{E}}$ therefore depends only on the total change in χ_1; this confirms the statement, made earlier in this section, that 20.4 is not sufficient to guarantee the validity of 20.3. Similar remarks obviously apply to the higher moments.

The reason for the result 20.21 is not hard to see. If a particle loses energy at different rates depending on the energy $\mathcal{E}$ which it has already lost, then this effect alone produces a "stretching" or a "bunching" of the distribution, depending on whether $\frac{\partial \chi_1}{\partial \mathcal{E}}$ is positive or negative. In the equations 20.13, whose solution is given by 20.2 and 20.3, this effect is completely neglected, even if we agree to evaluate the moments χ_1, χ_2, etc., at the mean value $\bar{\mathcal{E}}(x)$. if we think of a moment $\chi_n(\mathcal{E},x)$, for a given point x, as plotted against $\mathcal{E}$, then the solutions 20.3 correspond to replacing this curve by the horizontal straight line $\chi_n(\bar{\mathcal{E}}(x),x)$, which is independent of $\mathcal{E}$. We have seen that this requires the condition 20.5. If we keep also the first derivatives of the moments in 20.15 - 20.18, (terms in round brackets), we are effectively replacing the curve $\chi_n(\mathcal{E},x)$ vs $\mathcal{E}$ at each point

x by a straight line tangent to this curve at $\bar{\mathcal{E}}(x)$. The "stretching (or bunching) effect" is then taken into account to a first approximation, and we may expect the solution to be correct so long as practically all the particles stay within a region of the $\mathcal{E}$ scale in which a linear approximation to $\chi_n(\mathcal{E},x)$ vs $\mathcal{E}$ is valid; i.e. the condition 20.4 should now be sufficient. This solution will begin to fail only when the terms in square brackets in 20.15 - 20.18 become important.

21. Integration of the Equations of Motion.

Equations 20.15 - 20.18 must be solved as an infinite set of simultaneous differential equations, if we keep all the bracketed terms. However, if we neglect all terms involving higher than first derivatives of the moments of the collision spectrum, the equations can be solved one by one. According to the last paragraph of Section 20, this should give a good approximation to the moments so long as $\sigma_u/\bar{E}_k$ is small. We therefore drop all terms in square brackets in 20.15 - 20.18.

20.15 now becomes:

$$\frac{d\bar{\mathcal{E}}}{dx} = \chi_1(\bar{\mathcal{E}}(x), x) \qquad 21.1$$

In the general case, 21.1 is equivalent to 20.2, when we add the boundary condition:

$$\bar{\mathcal{E}}(0) = 0 \qquad 21.2$$

In case χ_1 does not depend on x explicitly, the solution of 21.1, 21.2 can be written:

$$x = \int_0^{\bar{\mathcal{E}}} \frac{d\mathcal{E}}{\chi_1(\mathcal{E})} \qquad 21.3$$

In any case, the function $\bar{\mathcal{E}}(x)$ is now determined, and the moments χ_n and their derivatives in the equations of motion for the moments $\overline{(\Delta\mathcal{E})^n}$ become functions of x only.

We have now to solve equation 20.16 in the form:

$$\frac{d\overline{(\Delta\mathcal{E})^2}}{dx} = 2\overline{(\Delta\mathcal{E})^2}\frac{\partial\chi_1}{\partial\mathcal{E}} + \chi_2 \;,\; \left(\overline{(\Delta\mathcal{E})^2} = 0 \text{ at } x = 0\right) \qquad 21.4$$

The solution of this inhomogeneous linear equation can be obtained by the usual methods:

$$\overline{(\Delta\mathcal{E})^2} = e^{2\int_0^x \frac{\partial\chi_1}{\partial\mathcal{E}}dx'} \int_0^x \chi_2 e^{-2\int_0^{x''} \frac{\partial\chi_1}{\partial\mathcal{E}}dx'} dx'' \qquad 21.5$$

where $\frac{\partial\chi_1}{\partial\mathcal{E}}$ and χ_2 are to be evaluated at $(\bar{\mathcal{E}}(x'), x')$ and $(\bar{\mathcal{E}}(x''), x'')$, respectively. If χ_1 does not depend explicitly on x, we have, by 21.1:

$$\int_0^x \frac{\partial\chi_1}{\partial\mathcal{E}}dx = \int_0^{\bar{\mathcal{E}}} \frac{1}{\chi_1}\frac{\partial\chi_1}{\partial\mathcal{E}}d\mathcal{E} = \log\frac{\chi_1(\bar{\mathcal{E}})}{\chi_1(0)} \qquad 21.6$$

21.5 now becomes:

$$\begin{aligned}\overline{(\Delta\mathcal{E})^2} &= \chi_1^2\int_0^x \frac{\chi_2}{\chi_1^2}dx \\ &= \chi_1^2(\bar{\mathcal{E}})\int_0^{\bar{\mathcal{E}}} \frac{\chi_2(\mathcal{E})}{\chi_1^3(\mathcal{E})}d\mathcal{E} \qquad 21.7\end{aligned}$$

21.7 is an obvious generalization of the corresponding equation 20.3.

$\overline{(\Delta\mathcal{E})^2}$ is now determined as a function of x, and 20.17 becomes:

$$\frac{d\overline{(\Delta\mathcal{E})^3}}{dx} = 3\,\overline{(\Delta\mathcal{E})^3}\frac{\partial\chi_1}{\partial\mathcal{E}} + \chi_3 + 3\overline{(\Delta\mathcal{E})^2}\,\frac{\partial\chi_2}{\partial\mathcal{E}} \qquad 21.8$$

The solution is:

$$\overline{(\Delta\mathcal{E})^3} = e^{3\int_0^x \frac{\partial\chi_1}{\partial\mathcal{E}}dx'} \int_0^x \left(\chi_3 + 3\,\overline{(\Delta\mathcal{E})^2}\frac{\partial\chi_2}{\partial\mathcal{E}}\right) e^{-3\int_0^{x''} \frac{\partial\chi_1}{\partial\mathcal{E}}dx'} dx'' \qquad 21.9$$

where $\frac{\partial \mathcal{X}_1}{\partial \varepsilon}$ is evaluated at $(\bar{\varepsilon}(x'), x')$, and $(\mathcal{X}_3 + 3\overline{(\Delta\varepsilon)^2}\frac{\partial \mathcal{X}_2}{\partial \varepsilon})$, at $(\bar{\varepsilon}(x''), x'')$. If $\mathcal{X}_1$ does not depend explicitly on x, this becomes:

$$\overline{(\Delta\varepsilon)^3} = \mathcal{X}_1^3 \int_0^x \frac{\mathcal{X}_3 + 3\overline{(\Delta\varepsilon)^2}\frac{\partial \mathcal{X}_2}{\partial \varepsilon}}{\mathcal{X}_1^3}\, dx$$

$$= \mathcal{X}_1^3(\bar{\varepsilon}) \int_0^{\bar{\varepsilon}} \frac{\mathcal{X}_3 + 3\overline{(\Delta\varepsilon)^2}\frac{\partial \mathcal{X}_2}{\partial \varepsilon}}{\mathcal{X}_1^4}\, d\varepsilon \qquad 21.10$$

The integrand in the last integral is evaluated at ε. Solutions of the equations for the higher moments are similar in principle, but more complicated. We shall not carry the procedure any farther.

Hereafter, all moments of the collision spectrum will be written as functions of the kinetic energy E_k or the total energy E instead of the energy loss ε, and we will assume the collision spectrum does not depend on x explicitly. We make the replacements:

$$\bar{\varepsilon} = E^0 - \bar{E} = E_k^0 - \bar{E}_k$$

$$\frac{\partial \mathcal{X}_n}{\partial \varepsilon} = -\frac{\partial \mathcal{X}_n}{\partial E} = -\frac{\partial \mathcal{X}_n}{\partial E_k} \qquad 21.11$$

The moments of the collision spectrum 8.1 are given by 14.17. We write the first three most conveniently in the form:

$$\frac{\mathcal{X}_1}{2\mu_e} = \frac{2}{\beta^2}\left[\log\frac{2\mu_e}{I_z} + \log(w^2-1) - \beta^2 - \frac{1}{2}\log(1+0.01\gamma w)\right]$$

$$\frac{\mathcal{X}_2}{(2\mu_e)^2} = \frac{1}{2}(w^2+1)\frac{1}{(1+0.01\gamma w)} \qquad 21.12$$

$$\frac{\mathcal{X}_3}{(2\mu_e)^3} = \frac{1}{6}(w^2-1)(w^2+2)\frac{1}{(1+0.01\gamma w)^2}$$

We define the functions:

$$f_1(E_k) = \gamma\int_0^{E_k}\frac{dE_k}{\chi_1(E_k)} = 50\int_1^W\frac{dw}{(\chi_1/2\mu_e)} \qquad \left(\gamma = \frac{200\mu_e}{\mu}\right) \tag{21.13}$$

$$f_2(E_k) = \gamma\int_0^{E_k}\frac{\chi_2}{\chi_1^3}dE_k = 50\int_1^W\frac{\chi_2/(2\mu_e)^2}{(\chi_1/2\mu_e)^3}dw \tag{21.14}$$

$$f_{3a}(E_k) = \gamma\int_0^{E_k}\frac{\chi_3}{\chi_1^4}dE_k = 50\int_1^W\frac{\chi_3/(2\mu_e)^3}{(\chi_1/2\mu_e)^4}dw \tag{21.15}$$

$$g_3(E_k) = 3\int_0^{E_k}\frac{1}{\chi_1^2}\frac{\partial\chi_2}{\partial E}dE_k = 3\int_1^W\frac{1}{(\chi_1/2\mu_e)^2}\frac{\partial}{\partial w}\left(\frac{\chi_2}{(2\mu_e)^2}\right)dw \tag{21.16}$$

$$f_{3b}(E_k) = \gamma\int_0^{E_k}g_3\frac{\chi_2}{\chi_1^3}dE_k = 50\int_1^W g_3\frac{\chi_2/(2\mu_e)^2}{(\chi_1/2\mu_e)^3}dw \tag{21.17}$$

$$f_3(E_k) = f_{3a}(E_k) - f_{3b}(E_k) \tag{21.18}$$

These are all very rapidly increasing functions of E_k. (See Figs. 6-9.)

In terms of the function $f_1(E_k)$, we can write 21.3 in the form:

$$f_1(\bar{E}_k) = f_1(E_k^0) - \gamma x \tag{21.19}$$

21.19 determines the mean energy $\bar{E}_k$ in terms of the initial kinetic energy E_k^0 and the distance x traversed by the particle. The mean range (in collision units) of particles with initial energy E_k^0 is, by 21.19, setting $\bar{E}_k = 0$:

$$\bar{R} = \frac{1}{\gamma}f_1(E_k^0) \tag{21.20}$$

Using 21.14, equation 21.7 becomes:

$$\overline{(\Delta\mathcal{E})^2} = \chi_1^2(\bar{E}_k)\frac{1}{\gamma}\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right] \qquad 21.21$$

or:

$$\sigma_u = \chi_1(\bar{E}_k)\gamma^{-1/2}\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right]^{1/2} \qquad 21.22$$

So long as $\bar{\mathcal{E}}$ is small enough so that $\chi_1(\bar{E}_k) \doteq \chi_1(E_k^\circ)$, 21.21 will agree with the formula 14.23 of Chapter III. When $\bar{\mathcal{E}}$ is moderately large, so that $f_2(\bar{E}_k) << f_2(E_k^\circ)$, 21.22 behaves like 20.21, and further changes in σ_u are due entirely to the "stretching" (or bunching) effect of variations in χ_1 with E_k.

Using 21.15, 21.16, and 21.21, equation 21.10 becomes:

$$\overline{(\Delta\mathcal{E})^3} = \chi_1^3(\bar{E}_k)\frac{1}{\gamma}\left\{f_{3a}(E_k^\circ) - f_{3a}(\bar{E}_k) - \int_{\bar{E}_k}^{E_k^\circ}\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right]\frac{dg_3}{d\bar{E}_k}\,d\bar{E}_k\right\} \qquad 21.23$$

Integrating the second term in the integrand by parts, and using 21.14 and 21.17:

$$-\int_{\bar{E}_k}^{E_k^\circ}\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right]\frac{dg_3}{d\bar{E}_k}\,d\bar{E}_k = g_3(\bar{E}_k)\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right] - f_{3b}(E_k^\circ) + f_{3b}(E_k) \qquad 21.24$$

Using 21.24 and 21.18, 21.23 becomes:

$$\overline{(\Delta\mathcal{E})^3} = \chi_1^3(\bar{E}_k)\frac{1}{\gamma}\left\{f_3(E_k^\circ) - f_3(\bar{E}_k) + g_3(\bar{E}_k)\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right]\right\} \qquad 21.25$$

The unweighted skewness is, by 21.22 and 21.25:

$$\lambda_u = \frac{\overline{(\Delta\mathcal{E})^3}}{\sigma_u^3} = \gamma^{1/2}\left\{\frac{f_3(E_k^\circ) - f_3(\bar{E}_k)}{\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right]^{3/2}} + \frac{g_3(\bar{E}_k)}{\left[f_2(E_k^\circ) - f_2(\bar{E}_k)\right]^{1/2}}\right\} \qquad 21.26$$

The second term is ordinarily a small correction. It turns out to be negligible when $\frac{E_k}{\mu} < 0.1$. It may amount to 25% of the first term for small thicknesses x when $\frac{E_k}{\mu} \sim 10$, 21.26 will agree with 14.24 for small values of $\bar{E}$. For moderately large $\bar{E}$, 21.26 reduces to:

$$\lambda_u = \gamma^{1/2} \frac{f_3(E_k^0)}{[f_2(E_k^0)]^{3/2}} \qquad (\bar{E}_k << E_k^0) \qquad 21.27$$

Thus, provided our approximations are valid to this point, the curve approaches a limiting shape which is not Gaussian, but has a definite skewness depending on the initial energy. We will see later that this limiting value is about 0.3 to 0.5 for mesons ($\gamma = 1$) of 1 to 1000 Mev. When 21.27 is valid, random variations in energy loss are no longer effective in changing the shape of the curve. We see that the "stretching effect" alone will not change the shape of the curve, since λ_u in 21.27 is constant. This is to be expected, since we have kept only linear terms in ΔE in the expansion of $\chi_1(E_k)$, which corresponds to a uniform stretching or compressing along the E scale. This does not mean that the "stretching effect" does not influence the shape of the curve. When $E_m' << \sigma_u$, the fluctuations due to the random nature of the energy losses E' can be neglected. When the collision spectrum is constant, σ_u increases slowly enough so that these random fluctuations have time to establish a Gaussian distribution before they become ineffective. But the decrease of E_m' with decreasing E_k, and the increase of σ_u due to "stretching", ($\frac{\partial \chi_1}{\partial E_k} < 0$), both result in the condition $E_m' << \sigma_u$ before the Gaussian distribution has become established. Conversely, a "bunching effect" ($\frac{\partial \chi_1}{\partial E_k} > 0$) would tend to cancel the effect of the decrease of E'_m with E_k and promote the growth of a Gaussian distribution. Actually, (see Fig. 5a), $\frac{\partial \chi_1}{\partial E}$ is negative (stretching) for $\frac{E_k}{\mu} < 1$, and is very slightly positive for $\frac{E_k}{\mu} > 1.5$.

The quantity $\chi_1/2\mu_e$ (Eq. 21.12) is a function of the energy E_k, the ratio $2\mu_e/I_z$ which depends on the material traversed, and the ratio $\gamma = \frac{200\mu_e}{\mu}$. For mesons or heavier particles ($\gamma \leq 1$), the term $-\frac{1}{2}\log(1+0.01\gamma w)$ is only a small correction for $\frac{E_k}{\mu} \leq 10$; we will omit it in calculations in the present section, and derive appropriate correction terms in Section 22. The average ionization potential I_z is given approximately by (4,p.245); see Section 5:)

$$I_z \doteq 13.5\, Z \text{ ev} \qquad 5.7$$

We adopt a standard value $I_z = 330$ ev, which corresponds roughly to iron, and define the ratio:

$$q = \frac{\log(3000) + \log(w^2-1) - \beta^2}{\log\frac{2\mu_e}{I_z} + \log(w^2-1) - \beta^2} \qquad 21.28$$

$$\left(\frac{2\mu_e}{330\text{ev}} = 3000\right)$$

The ratio q, for various values of I_z, and the function $\chi_1/2\mu_e$ for $I_z = 330$ ev, are plotted vs $w_k = E_k/\mu$ in Figure 5. q is a slowly varying function of w_k. We can write χ_1 for any I_z in terms of χ_1 for $I_z = 330$ ev:

$$\chi_{1(I_z)} = \frac{1}{q}\,\chi_{1(330)} \qquad 21.29$$

If we neglect the factor $(1 + 0.01\gamma w)$, the quantities $\gamma_2/(2\mu_e)^2$ and $\gamma_3/(2\mu_e)^3$ are functions of w alone.

Several methods of integration were tried in evaluating the integrals 21.13 - 21.17, including the one used by Rossi and Greisen (4) in their computation of the integral 21.13; which consisted in performing the integration with $\log(w^2-1)$ regarded as constant, and using the resulting formula to build up the integral in steps small enough so that $\log(w^2-1)$ is practically constant over the range of w included in each step. The most satisfactory method, both for convenience in using a desk computing machine, and as a uniform procedure for all the integrations, was not found until after many of the calculations had been performed in other ways, a common misfortune. This method was to make a change of variable:

$$y = \log_{10} w_k$$

$$dw = w_k \log_e 10 \, dy = 2.303\, w_k dy \qquad 21.30$$

and integrate over the variable y, using Simpson's rule with intervals of 0.1:

$$\int_{y_0}^{y_0+0.2} f(y)\,dy \doteq \frac{f(y_0) + 4f(y_0+0.1) + f(y_0+0.2)}{30} \qquad 21.31$$

The rule 21.31 was tested by computing $\int_{y_0}^{y_0+0.2} e^{\alpha y} dy$. The error is less than ½% for $|\alpha| \leq 10$, and less than 2.3% for $|\alpha| \leq 15$. Hence for an integrand proportional to $w_k^n dw_k \sim e^{2.3(n+1)y} dy$, the error will be less than ½% if $-5.3 \leq n \leq 3.3$. This condition is satisfied by the integrands in 21.13, 21.14, 21.16. For an error less than 2.3%, the condition is $-7.5 \leq n \leq 5.5$.

This condition is satisfied by the integrands in 21.15 and 21.17. The functions $\chi_{1(330)}/2\mu_e$, $\chi_2/(2\mu_e)^2$, $\chi_3/(2\mu_e)^3$ are tabulated vs w_k at 0.1 intervals of y in Table 7. The functions f_1, f_2, f_3, g_3, computed by numerical integration using 21.31, are tabulated vs w_k at 0.2 intervals of y in Table 8, and plotted in Figures 6,7,8,9. f_1 was computed for $I_2 = 100$ ev, 330 ev, and 1000 ev; f_2, f_3, and g_3 were computed for $I_2 = 330$ ev. The integrals were computed from the lower limit $w_k = 0.01$, and the values of the functions at $w_k = 0.01$ chosen to preserve the linearity of the curves when plotted logarithmically as in Figures 6 - 9. For $w_k \ll 1$, these integrals can be expressed in terms of the exponential integral (Eq. 16.3), which in turn can be expanded in an asymptotic series (Eq.16.6); This method was used as a check on the results for small values of w_k.

Since the integrands in 21.13 -- 21.17 are increasing functions of w, the major contribution to the integral comes from values of w near the upper limit. Since the factor q (Eq. 21.28) is a slowly varying function of w, it can be taken outside the integral and evaluated at or near the upper limit w. We therefore have:

$$
\begin{aligned}
f_1 &\doteq q f_{1(330)} \\
f_2 &\doteq q^3 f_{2(330)} \\
f_3 &\doteq q^4 f_{3(330)} \\
g_3 &\doteq q^2 g_{3(330)}
\end{aligned}
\qquad 21.32
$$

For f_2, f_3, g_3, the q in these formulas should be evaluated at about 80% of the upper limit of the integral. For f_1, q should be evaluated at about one half the upper limit.

The computation of f_1 was checked against the range-energy relation for protons up to 15 Mev ($w_k = 0.016$) given by Bethe and Livingston (10,p.269), and against the range-momentum relation for mesons above $w_k = 0.5$ given by Rossi and Greisen (4,p.249). The complete range-energy relation for mesons of about $w_k = 0.01$ to 2.5 is given in an article by Wheeler and Ladenburg (33).

By means of the curves given in Figures 5 - 9, and the formulas 21.19, 21.22, 21.26, and 21,32, $\bar{E}$, σ_u, and λ_u can be determined for any type of material and any initial energy E_k^0 and thickness x such that w_k^0 and $\bar{w}_k$ are between 0.01 and 10, subject to small corrections $O(\frac{\gamma w}{100})$ and an upper limit on x to be determined in the next section.

22. Corrections and Limits of Validity.

We will compute the first order corrections due to neglect of the factor $(1 + 0.01\gamma w)$ in the preceding section. For mesons, these corrections are negligible for $w_k < 1$ $(w < 2)$, and we are only interested in values of $w_k < 10$. For protons $(\gamma \sim 0.1)$, the corrections can be neglected entirely for $w_k < 10$. By 21.12, we have

$$\frac{\mathcal{K}_1}{2\mu_e} = \frac{2}{\beta^2}\left[\log\frac{2\mu_e}{I_2} + \log(w^2-1) - \beta^2 - \frac{1}{2}\log(1+0.01\gamma w)\right]$$

$$\doteq \frac{2}{\beta^2}\left[\log\frac{2\mu_e}{I_2} + \log(w^2-1 - \beta^2\right]\left[1 - \frac{\gamma w/200}{\log\frac{2\mu_e}{I_2}(w^2-1) - \beta^2}\right] \qquad 22.1$$

The denominator in the correction factor varies from 8.4 to 11.8 as w_k varies from 1 to 10, for $I_z = 330$ ev. Variations in I_z from 100 ev (air) to 1000 ev (lead) will introduce an additional change in the denominator of ± 1. We may take 10 as a sufficiently close value, and write 22.1 as:

$$\frac{\chi_1}{2\mu_e} \doteq \frac{2}{\beta^2}\left[\log\frac{2\mu_e}{I_z} + \log(w^2-1) - \beta^2\right]\left[1 - \frac{\gamma w}{2000}\right] \qquad 22.2$$

The correction here is entirely negligible, amounting to only ½% at $w = 10$. The correction to f_1 is therefore negligible.

Using the symbol "$\sim$" to indicate the dependence of a quantity on terms $O(\gamma w)$, we have, by 21.12:

$$\frac{\chi_2}{(2\mu_e)^2} \sim \frac{1}{1+0.01\gamma w} \sim \left(1 - \frac{\gamma w}{100}\right) \qquad 22.3$$

$$\frac{\chi_3}{(2\mu_e)^3} \sim \frac{1}{(1+0.01\gamma w)^2} \sim \left(1 - \frac{\gamma w}{50}\right) \qquad 22.4$$

Using 22.2, 22.3, 22.4, 21.14 - 21.18, we find:

$$f_2 \sim \frac{\left(1 - \frac{\gamma w}{100}\right)}{\left(1 - \frac{\gamma w}{2000}\right)^3} \sim (1 - 0.007\gamma w) \qquad 22.5$$

$$f_3 \sim \frac{\left(1 - \frac{\gamma w}{50}\right)}{\left(1 - \frac{\gamma w}{2000}\right)^4} \sim (1 - 0.015\gamma w) \qquad 22.6$$

The corrections on the right have been reduced about 20% to compensate for the fact that the major contribution to the integrals f_2 and f_3 comes from values of w averaging about 80% of the upper limit.

From 22.5 and 22.6, we can estimate the correction to be applied to λ_u when computed from the uncorrected f_2 and f_3. (Corrections in g_3 need not be considered, since the term in g_3 is itself a small correction.)

$$\lambda_u \sim \frac{f_3}{f_2^{3/2}} \sim (1-0.005\gamma w) \qquad 22.7$$

This correction is practically negligible, amounting to only 5% at $w_k=10$. The correction to σ_u is also practically negligible, amounting to about 4% at $w_k=10$:

$$\sigma_u \sim \chi_1 f_2^{1/2} \sim (1-0.004\gamma w) \qquad 22.8$$

We now consider the errors introduced by the neglect of the terms in square brackets in 20.15 - 20.17. The variation in χ_1 with E_k is only important for small values of β^2, and we can estimate the order of magnitude of the error by taking χ_1 to be proportional to E_k^{-1}:

$$\chi_1 \sim E_k^{-1} \qquad 22.9$$

$$\frac{1}{2}\overline{(\Delta E)^2}\,\frac{1}{\chi_1}\frac{\partial^2\chi_1}{\partial E^2} \doteq \left(\frac{\sigma_u}{E_k}\right)^2 \qquad 22.10$$

22.10 gives the (first order) percentage correction to the right side of equation 21.1. If we take as the limit of validity of equation 21.1 the point where the correction becomes 10% of the main term, then we should regard 21.1 as valid until σ_u becomes 30% of $\overline{E_k}$.

Proceeding similarly, we find for the first order percentage corrections to the right side of 21.4, using 21.12:

$$\frac{1}{4\frac{\partial\chi_1}{\partial E}}\,\frac{\partial^2\chi_2}{\partial E^2} \doteq 0 + O\left(\frac{\gamma w}{100}\right) \qquad 22.11$$

$$\frac{1}{2}\,\frac{\overline{(\Delta\varepsilon)^3}}{\overline{(\Delta\varepsilon)^2}}\,\frac{1}{\frac{\partial\mathcal{K}_1}{\partial\varepsilon}}\,\frac{\partial^2\mathcal{K}_1}{\partial\varepsilon^2} \doteq \lambda_u\,\frac{\sigma_u}{E_k} \doteq \frac{1}{2}\,\frac{\sigma_u}{E_k} \qquad 22.12$$

If we require that the correction be no more than 10%, then 21.4 is valid until σ_u is 20% of $\bar{E}_k$. Here we have compared the terms in square brackets with the term in round brackets, on the assumption that the latter is the important term. This is readily verified by comparing the first two terms in 21.4:

$$\frac{2\,\overline{(\Delta\varepsilon)^2}}{\mathcal{K}_2}\,\frac{\partial\mathcal{K}_1}{\partial\varepsilon} \doteq 2\,\frac{\mathcal{K}_1 E_k}{\mathcal{K}_2}\left(\frac{\sigma_u}{E_k}\right)^2 \doteq 4000\,\frac{1}{\gamma}\,\frac{w^2}{(w+1)(w^2+1)}\left(\frac{\sigma_u}{E_k}\right)^2$$

$$\doteq \frac{1000}{\gamma}\left(\frac{\sigma_u}{E_k}\right)^2 \quad (w \doteq 1) \qquad 22.13$$

Thus the second term in 21.4 becomes equal to the first when $\sigma_u \doteq \bar{E}_k/30$, and for greater thicknesses the second term is dominant. We see that with the same criterion as was used above, equation 20.3 can be regarded as valid only until σ_u becomes 1% of $\bar{E}_k$, when $w_k \ll 1$. When $w_k = 10$, 20.3 is valid until σ_u is about 3% of $\bar{E}_k$.

Taking $3\,\overline{(\Delta\varepsilon)^3}\,\frac{\partial\mathcal{K}_1}{\partial\varepsilon}$ as the important term in 20.17, and assuming $\overline{(\Delta\varepsilon)^4} = 3\,\overline{(\Delta\varepsilon)^2}^2$, which holds for a Gaussian distribution, we find:

$$\frac{1}{2}\,\frac{\overline{(\Delta\varepsilon)^4} - \overline{(\Delta\varepsilon)^2}^2}{\overline{(\Delta\varepsilon)^3}}\,\frac{1}{\frac{\partial\mathcal{K}_1}{\partial\varepsilon}}\,\frac{\partial^2\mathcal{K}_1}{\partial\varepsilon^2} \doteq \frac{1}{\lambda_u}\,\frac{\sigma_u}{E_k} \doteq 2\,\frac{\sigma_u}{E_k} \qquad 22.14$$

$$\frac{1}{2}\,\frac{1}{\frac{\partial\mathcal{K}_1}{\partial\varepsilon}}\,\frac{\partial^2\mathcal{K}_2}{\partial\varepsilon^2} \doteq 0 + O\left(\frac{\gamma w}{100}\right) \qquad 22.15$$

$$\frac{3}{2}\,\frac{\overline{(\Delta\varepsilon)^2}}{\overline{(\Delta\varepsilon)^3}}\,\frac{1}{\frac{\partial\mathcal{K}_1}{\partial\varepsilon}}\,\frac{\partial^2\mathcal{K}_3}{\partial\varepsilon^2} \doteq \frac{\gamma^2}{10^5}\,2\,w_k\,\frac{E_k}{\sigma_u} \quad (w_k \ll 1) \qquad 22.16$$

Only 22.14 gives a restriction on σ_u/E_k. For a 10% correction, we must require $\sigma_u \leq 0.05\bar{E}_k$. If we allow σ_u to be 20% of $\bar{E}_k$, the correction term is about 40%. Actually it can be shown by writing an equation for $d\lambda_u/dx$ that the corrections 22.14 and 22.13 partially cancel each other, so that the correction to the λ_u equation is only about one fourth of 22.14.

We will consider $\bar{E}$, σ_u, λ_u, as computed by the method given in Section 21, as correct for:

$$\sigma_u \leq \frac{1}{5}\bar{E}_k \qquad 22.17$$

or:

$$\frac{\gamma^{1/2}}{20}\,\frac{\chi_1}{2\mu_e}\,\frac{(f_2(E_k^0))^{1/2}}{\bar{w}_k} \leq 1 \qquad 22.18$$

where $f_2(\bar{E}_k)$ has been neglected in comparison with $f_2(E_k^0)$. Using Figures 5 and 7, we find that the results of Section 21 can be trusted until the energy loss and the path length have reached the limiting values shown in the following table ($\gamma = 1$).

Limit of Validity of Formulas for Energy Loss Distribution Parameters

E_k^0/μ	$\bar{E}/E_k^0$	% of range traversed
10	85%	86%
1	80%	87%
0.1	72%	88%
0.033	67%	88%

For any E_k^0/μ between 10 and 0.01, we can regard the formulas of Section 21 as valid for thickness of material traversed up to 85% of the range of the particles. It may be of interest that at this limiting thickness, $f_2(\bar{E}_k)$ is in all cases less than 2% of $f_2(E_k^0)$.

The energy loss distribution curves determined by the methods presented will begin to fail first in their tails, where non-linear "stretching" due to the positive value of $\frac{\partial^2 \chi_1}{\partial E^2}$ becomes important. From the method of derivation of equations 21.1, 21.4, 21.8, we can conclude that the distribution curves thereby predicted will always agree with the correct distribution in the neighborhood of the predicted mean $\bar{\mathcal{E}}$, but the region of agreement will become a smaller and smaller fraction of the total spread in the distribution as the limits of validity are further exceeded. Since, at thicknesses x for which the formulas of Section 21 break down, only the terms in χ_1 and its derivatives are important in equations 20.15 - 20.18, we could use the solution 9.15 of equation 9.11 to extend the distribution curves to larger values of x. This gives essentially a complete solution of the problem of determining $P(\mathcal{E},x)$ for mesons or heavier particles at practically all values of x. However, it is unlikely that the curve $P(\mathcal{E},x)$ is of much interest for thicknesses of material greater than about 85% of the range. For greater thicknesses, the distribution in range is of more interest; this problem is to be treated in the next chapter.

In conclusion, we may state more precisely the limits of validity for the formulas derived in Chapter III. As a measure of the error in neglecting the variation of $\chi(E')$ with ε, we take the ratio of the first two terms in 21.4, which is, by 22.13:

$$2\,\frac{\overline{(\Delta\varepsilon)^2}}{\chi_2}\,\frac{\partial \chi_1}{\partial \varepsilon} \doteq 2\,\frac{\chi_1 E_k}{\chi_2}\left(\frac{\sigma_u}{E_k}\right)^2 \qquad 22.13$$

Multiplying numerator and denominator by x, we arrive at the approximate formula:

$$2\,\frac{\overline{(\Delta\varepsilon)^2}}{\chi_2}\,\frac{\partial \chi_1}{\partial \varepsilon} \doteq 2\,\frac{\bar{\varepsilon}}{E_k} \qquad 22.19$$

Requiring as usual that this ratio be 10% or less, we find that $\bar{\varepsilon}$ must be not more than 5% of E_k^0. Since $[f_2(E_k^0) - f_2(\bar{E}_k)]$ can hardly be determined from Figure 7 to within better than 20% for such small values of $\bar{\varepsilon}/E_k^0$, it is probably satisfactory to use Figure 3 to get the weighted parameters directly when $\bar{\varepsilon}$ is less than 10% of E_k^0. 22.19 applies only at low energies, since we assumed β^2 proportional to E_k in estimating $\frac{\partial \chi_1}{\partial \varepsilon}$. At higher energies, the ratio given by 22.19 is too large, and Figure 3 can be used to greater thicknesses x than allowed by the above rule. By determining the values of λ_u at various values of x and E_k^0, it has been found that Figure 3 can be used whenever λ_u or λ_w is about 1 or greater. This gives a less restrictive condition at higher energies; at lower energies, the condition $\bar{\varepsilon}/E_k^0 \lesssim \frac{1}{10}$ should be used.

CHAPTER V

RELATED PROBLEMS, CONCLUSION

23. Distribution in Path Length and Range.

Related to the distribution $P(\mathcal{E},x)$ is the distribution $Q(x,\mathcal{E})$ defined by requiring that $Q(x,\mathcal{E})dx$ be the probability that, in losing a total amount of energy $\mathcal{E}$, a particle will traverse a distance between x and $x+dx$. Let us consider a large number n of particles which start at $x=0$ with a common initial energy E^0. We assume that the energy loss $\mathcal{E}$ for each particle is some monotonically increasing function of the path length x. The number of particles which lose an energy $\mathcal{E}$ or less in going the distance x is:

$$n\int_0^{\mathcal{E}} P(\mathcal{E},x)d\mathcal{E}$$

The number which lose an energy greater than $\mathcal{E}$ in going a distance x is the number which lose an amount $\mathcal{E}$ in going a distance less than x, i. e., it is:

$$n\int_0^{x} Q(x,\mathcal{E})dx$$

The sum of these two numbers is n, so we have:

$$\int_0^{\mathcal{E}} P(\mathcal{E},x)d\mathcal{E} + \int_0^{x} Q(x,\mathcal{E})dx = 1 \qquad 23.1$$

Differentiating 23.1, we arrive at the formulas:

$$Q(x,\mathcal{E}) = -\int_0^{\mathcal{E}} \frac{\partial}{\partial x} P(\mathcal{E},x)d\mathcal{E} \qquad 23.2$$

$$P(\mathcal{E},x) = -\int_0^{x} \frac{\partial}{\partial \mathcal{E}} Q(x,\mathcal{E})dx \qquad 23.3$$

$$\frac{\partial Q}{\partial \varepsilon} + \frac{\partial P}{\partial x} = 0 \qquad 23.4$$

These formulas give the connection between the two distributions provided only that, for each particle, $\varepsilon(x)$ is a single valued monotonically increasing function of x. When this is not the case, special conventions must be made as to the meaning of the functions $P(\varepsilon,x)$ and $Q(x,\varepsilon)$; if they are defined as the excess of the fractional number of particles reaching the given value of ε (or x) from a smaller value over the number reaching the given value from a larger value, then the formulas 23.1 - 23.4 are still valid.

The formulas 23.2 and 23.3 are rather cumbersome to apply in general for the calculation of $Q(x,\varepsilon)$ from $P(\varepsilon,x)$ or vice versa. We shall apply them here when certain conditions are satisfied. $P(\varepsilon,x)$ can be written (e.g. 18.7) in the form:

$$P(\varepsilon,x) = \frac{1}{\sigma_u} P_{\lambda_u, \zeta_u, \ldots}(\Delta_u) \;, \quad \Delta_u = \frac{\varepsilon - \bar{\varepsilon}}{\sigma_u} \qquad 23.5$$

Here λ_u, ζ_u, ..., are dimensionless parameters which specify the shape of the curve. The parameters $\bar{\varepsilon}$, σ_u, λ_u, ζ_u, ..., are functions of x, but we assume that, in computing $\frac{\partial P}{\partial x}$, only the variation of $\bar{\varepsilon}$ with x need be taken into account:

$$-\frac{\partial}{\partial x} P(\varepsilon,x) = \frac{1}{\sigma_u^2} \frac{d\bar{\varepsilon}}{dx} \frac{\partial}{\partial \Delta_u} P_{\lambda_u, \zeta_u, \ldots}(\Delta_u) \qquad 23.6$$

The justification for this assumption will be indicated below. Substituting 23.6 in 23.2, using $d\varepsilon = \sigma_u d\Delta_u$:

$$Q(x,\varepsilon) = \frac{1}{\sigma_u} \frac{d\bar{\varepsilon}}{dx} P_{\lambda_u, \zeta_u, \ldots}(\Delta_u) \qquad 23.7$$

Defining $\overline{x}(\mathcal{E})$ and Δx by:

$$\overline{\mathcal{E}}(\overline{x}) = \mathcal{E}$$

$$\Delta x = x - \overline{x} \qquad 23.8$$

we obtain:

$$\Delta_u = -\frac{\Delta x}{\sigma_u}\frac{d\overline{\mathcal{E}}}{dx} + \frac{1}{2}\frac{(\Delta x)^2}{\sigma_u^2}\left(2\frac{d\overline{\mathcal{E}}}{dx}\frac{d\sigma_u}{dx} - \sigma_u\frac{d^2\overline{\mathcal{E}}}{dx^2}\right) + \cdots \qquad 23.9$$

where σ_u, $\frac{d\overline{\mathcal{E}}}{dx}$, etc., are evaluated at $\overline{x}$. Note that derivatives of σ_u occur only in the higher order terms, so that for small Δx, we are justified in neglecting the variation of σ_u with x. It can similarly be shown that the terms arising from the derivatives of the other parameters are incomparison with the right side of 23.6, of the order of the fractional change in these parameters when x changes by $\left(\overline{(\Delta x)^2}\right)^{1/2}$, so that when the spread in x values is small, the present approximation is justified. Neglecting higher order terms in 23.9, and defining:

$$\left(\overline{(\Delta x)^2}\right)^{1/2} = \sigma_u \Big/ \frac{d\overline{\mathcal{E}}}{dx} \qquad 23.10$$

we have, by 23.7, 23.9:

$$Q(x,\mathcal{E}) = \frac{1}{\left(\overline{(\Delta x)^2}\right)^{1/2}} P_{\lambda_u, \zeta_u, \ldots}\left(-\frac{\Delta x}{\left(\overline{(\Delta x)^2}\right)^{1/2}}\right) \qquad 23.11$$

Reversing the sign of the variable will reverse the signs of the odd parameters:

$$Q(x,\mathcal{E}) = \frac{1}{\left(\overline{(\Delta x)^2}\right)^{1/2}} P_{-\lambda_u, \zeta_u, \mu_u}\left(\frac{\Delta x}{\left(\overline{(\Delta x)^2}\right)^{1/2}}\right) \qquad 23.12$$

Comparing 23.12 with 23.5, we see that $\bar{x}$ and $\left(\overline{(\Delta x)^2}\right)^{1/2}$ are indeed the mean value and r.m.s. fluctuation of x for the distribution $Q(x,\mathcal{E})$, and the dimensionless parameters specifying the shape of the curve are the same as those of $P(\mathcal{E},x)$ except for a reversal of sign of the odd parameters corresponding to a reflection of the curve in the mean value. The parameters $-\lambda_u$, ζ_u, ..., for the curve $Q(x,\mathcal{E})$ are to be considered as functions of $\mathcal{E}$ determined by evaluating the corresponding parameters for $P(\mathcal{E},x)$ at $x = \bar{x}(\mathcal{E})$, with $\bar{x}$ given by 23.8; $\left(\overline{(\Delta x)^2}\right)^{1/2}$ is also a function of $\mathcal{E}$ determined in a similar way with the help of 23.10. Formula 23.12 (with 23.5) for $Q(x,\mathcal{E})$ in terms of $P(\mathcal{E},x)$ is valid so long as:

$$\frac{\left(\overline{(\Delta x)^2}\right)^{1/2}}{\sigma_u}\,\frac{d\sigma_u}{dx} = \frac{d\sigma_u}{d\mathcal{E}} << 1$$

with similar conditions on the other parameters. Using 21.1 and 21.4:

$$\frac{d\sigma_u}{d\mathcal{E}} = \frac{1}{2}\,\frac{\chi_2}{\sigma_u\,\chi_1} + \frac{\sigma_u}{\chi_1}\,\frac{d\chi_1}{\partial\mathcal{E}}$$

The second term is negligible whenever the fundamental assumption 20.4 of Chapter IV is valid. The first is important for small values of x. Setting $\sigma_u \doteq \chi_2 x$, the condition on $\frac{d\sigma_u}{dx}$ becomes:

$$\frac{1}{2}\,\frac{\chi_2^{1/2}}{\chi_1}\,\frac{1}{x^{1/2}} << 1 \qquad 23.13$$

$$\left(\text{or } \frac{1}{2}\,\frac{\sigma_u}{\mathcal{E}} << 1\right)$$

Thus our approximation breaks down only for fairly small distances, or for very large distances when the fundamental assumption 20.4 breaks down.

At sufficiently large values of x so that the term in $g_3(\bar{E}_k)$ can be neglected, we have, by 21.1, 21.22, 21.26:

$$\frac{d\bar{E}}{dx} = \chi_1(\bar{E}_k)$$

$$\sigma_u = \left(\overline{(\Delta E)^2}\right)^{1/2} = \chi_1(\bar{E}_k)\gamma^{-1/2}\left[f_2(E_k^0) - f_2(\bar{E}_k)\right]^{1/2}$$

$$\lambda_u = \frac{\overline{(\Delta E)^3}}{\left(\overline{(\Delta E)^2}\right)^{3/2}} = \gamma^{1/2}\frac{f_3(E_k^0) - f_3(\bar{E}_k)}{\left[f_2(E_k^0) - f_2(\bar{E}_k)\right]^{3/2}}$$

We use the results of the preceding paragraph together with these formulas and 21.19 to obtain:

$$\bar{x} = \frac{1}{\gamma}\left[f_1(E_k^0) - f_1(E_k^f)\right]$$

$$\overline{(\Delta x)^2} = \frac{1}{\gamma}\left[f_2(E_k^0) - f_2(E_k^f)\right] \qquad 23.14$$

$$\overline{(\Delta x)^3} = -\frac{1}{\gamma}\left[f_3(E_k^0) - f_3(E_k^f)\right]$$

where $E_k^f = E_k^0 - \mathcal{E}$. These formulas are valid when 23.13 holds, and provided E_k^f is not so small ($\bar{x}$ so large) that the method of Chapter IV becomes invalid. Let us consider an ionizing particle which, in traversing a medium, falls from an initial energy E_k^0 to a final energy E_k^f. Choose a set of energies E_k^0, E_k^1, E_k^2, . . . , E_k^f, spaced so that 23.14 applies to the distance $x_{n\rightarrow n+1}$ travelled by the particle in falling from an initial energy E_k^n to a final energy E_k^{n+1}. Evidently the distances

$x_{n\to n+1}$ travelled by the particle in losing the energies $E_k^n - E_k^{n+1}$ for various values of n are independent of one another,* if the collision spectrum does not depend explicitly on the distance x. Therefore, according to the additive property of semi-invariants (Sec.12), we can add the $\bar{X}$, $\overline{(\Delta x)^2}$, $\overline{(\Delta x)^3}$, for the several E_k intervals, and 23.14 holds in general except for the limitation 23.13.

We are particularly interested in the distribution in range, which is $Q(x, E = E_k^0)$. Setting $E_k^f = 0$, and writing R (range) for x, 23.14 gives:

$$\bar{R} = \frac{1}{\gamma} f_1(E_k^0)$$

$$\overline{(\Delta R)^2} = \frac{1}{\gamma} f_2(E_k^0) \qquad 23.15$$

$$\overline{(\Delta R)^3} = -\frac{1}{\gamma} f_3(E_k^0)$$

R in 23.15 is measured, like x, in collision units. The percentage fluctuation in range, and the skewness in range distribution are:

$$\frac{\left(\overline{(\Delta R)^2}\right)^{1/2}}{\bar{R}} = \gamma^{1/2} \frac{[f_2(E_k^0)]^{1/2}}{f_1(E_k^0)}$$

$$\frac{\overline{(\Delta R)^3}}{\left(\overline{(\Delta R)^2}\right)^{3/2}} = -\gamma^{1/2} \frac{f_3(E_k^0)}{[f_2(E_k^0)]^{3/2}} \qquad 23.16$$

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* This is only true when $E_k^n - E_k^{n+1}$ is large compared with the energy lost in a single collision. Otherwise adjacent energy intervals may be linked statistically by the fact that one collision can contribute an appreciable fraction of the energy loss in both intervals. This effect may be related to the breakdown of 23.14 for very small E (small $\bar{X}$).

These functions (for $Y=1$) are plotted in Figures 10a and 10b.

The formula 23.15 for $\overline{(\Delta R)^2}$ (and also for $\bar{R}$) was given essentially by Bohr (1), and has been used by Bethe and Livingston (10,p.283) to compute the percentage fluctuation in range. Bethe and Livingston plot $\sqrt{\frac{\pi}{2}}\,(\overline{(\Delta R)^2})^{1/2}/\bar{R}$ vs $\bar{R}$ for protons in air for ranges from 2 to 200 cm, or up to 13.6 Mev, which corresponds to $E_k/\mu = 0.0146$. Figure 10a agrees with the curve given in (10) for $E_k/\mu = 0.01$ to 0.015.

By using 23.14 to get $Q(x,\mathcal{E})$ when $\mathcal{E}$ and $\bar{X}$ are large, and then using 23.3, one could obtain in principle a solution for $P(\mathcal{E},x)$ at distances x for which the condition 20.4 of Chapter IV no longer holds. Formulas 23.2 and 23.3, however, require a prohibitive amount of labor in general when $\overline{(\Delta X)^2}$ and $\overline{(\Delta \mathcal{E})^2}$ are not small.

24. Distribution in Initial Energies.

We have determined in Chapters III and IV the distribution $P(\mathcal{E},x)$ of energy loss, given the initial energy E^0 and the distance X traversed by a particle. For purposes of this section, it will be convenient to include the argument E^0 explicitly* and write:

$P(\mathcal{E},E^0,x)\,d\mathcal{E}$ = probability that a particle with initial energy E^0 loses an energy between $\mathcal{E}$ and $\mathcal{E}+d\mathcal{E}$ in traversing the distance x.

It is often required to know the probability distribution of the initial energy E^0, when the final energy E^f and the distance

---

* We will use the symbol "E^0" instead of "E_k^0" to save writing a subscript; this does not affect the discussion.

x traversed are given. We will express this probability by the function:

$S(E^\circ; E^f, x)\,dE^\circ =$ probability that a particle which, after traversing a distance x, has a final energy E^f, had initially an energy between E° and $E^\circ + dE^\circ$. 24.1

Obviously $P(E, E^\circ; x)$ can be interpreted as a distribution in final energies:

$P(E^\circ - E^f, E^\circ; x)\,dE^f =$ probability that a particle with initial energy E° has, after traversing a distance x, a final energy between E^f and $E^f + dE^f$. 24.2

The lack of symmetry in the notations may serve to remind us that the two cases are not really the converses of each other in that the probability $S(E^\circ; E^f, x)\,dE^\circ$ is not determined by giving E^f and x, but we must also give the "a priori" probability distribution. This is the probability distribution in E° values before the value of E^f is known. We may know, for example, that the particle is one of a group of particles with a certain distribution in energies; we may know that it was observed in a certain piece of apparatus at a certain time; we may know that it has previously traversed a thickness x' of material having started originally with a known energy; etc. Whatever information of this sort we possess about E° is to be included in the function:

$A(E^\circ)\,dE^\circ =$ probability that a particle has an initial energy between E° and $E^\circ + dE^\circ$ when the final energy E^f is unknown. 24.3

Consider a large number n of particles whose initial energy distribution is given by the function $A(E^\circ)$. We have, by the above definitions:

$n A(E^\circ) dE^\circ P(E^\circ - E^f, E^\circ; x) \, dE^f =$ number of particles with an initial energy between E° and $E^\circ + dE^\circ$ and a final energy between E^f and $E^f + dE^f$. 24.4

The probability that a particle with final energy between E^f and $E^f + dE^f$ had an initial energy between E° and $E^\circ + dE^\circ$, is just the number of particles given by 24.4 divided by the total number of particles with final energies between E^f and $E^f + dE^f$. When dE^f is small, this probability is that defined in 24.1:

$$S(E^\circ; E^f, x) \, dE^\circ = \frac{n A(E^\circ) dE^\circ P(E^\circ - E^f, E^\circ; x) dE^f}{n \, dE^f \int_0^\infty A(E^\circ) P(E^\circ - E^f, E^\circ; x) dE^\circ} \qquad 24.5$$

This is an instance of Bayes' theorem (34,p.61). We thus get the connection between the functions S and P:

$$S(E^\circ; E^f, x) = \frac{A(E^\circ) P(E^\circ - E^f, E^\circ; x)}{\int_0^\infty A(E^\circ) P(E^\circ - E^f, E^\circ; x) dE^\circ} \qquad 24.6$$

If, as will usually be the case, $A(E^\circ)$ is constant over the (very small) range of initial energies for which there is an appreciable probability, 24.6 becomes:

$$S(E^\circ; E^f, x) = \frac{P(E^\circ - E^f, E^\circ; x)}{\int_0^\infty P(E^\circ - E^f, E^\circ; x) dE^\circ} \qquad 24.7$$

We now write the function $P(\varepsilon, E^\circ; x)$ in the form 23.5 used in the last section:

$$P(\varepsilon, E^\circ, x) = \frac{1}{\sigma_u} P_{\lambda_u, \zeta_u, \ldots}(\Delta_u) \quad , \quad \Delta_u \doteq \frac{\varepsilon - \bar{\varepsilon}}{\sigma_u} \qquad 24.8$$

The parameters $\bar{\varepsilon}$, σ_u, λ_u, ζ_u, ..., are functions of E° and x, but they will not change appreciably over a range of E° values small compared to E_k° itself, and we will assume that the values with appreciable probability are confined to such a small range. This assumption will be confirmed by the results, and evidently must hold if range measurements are to give any accurate measurement of initial energies. When E^f is given, we have:

$$E^0 = E^f + \mathcal{E} \qquad 24.9$$

$\bar{\mathcal{E}}$ is given by (21.3):

$$x = \int_{E^0 - \bar{\mathcal{E}}}^{E^0} \frac{dE}{K_1(E)} \qquad 24.10$$

We define $\overline{E^0}$ by the equation:

$$x = \int_{E^f}^{\overline{E^0}} \frac{dE}{K_1(E)} = \frac{1}{\gamma}\left[f_1(\overline{E^0}) - f_1(E^f)\right] \qquad 24.11$$

E^f and E^0 evidently satisfy the relation:

$$\overline{E^0} = E^f + \bar{\mathcal{E}}(\overline{E^0}, x) \qquad 24.12$$

We define ΔE^0 by:

$$\Delta E^0 = E^0 - \overline{E^0} = \mathcal{E} - \bar{\mathcal{E}}(\overline{E^0}, x) \qquad 24.13$$

The latter equality follows from 24.12 and 24.9. We now have:

$$\Delta_u = \frac{\mathcal{E} - \bar{\mathcal{E}}(E^0, x)}{\sigma_u}$$

$$= \frac{\mathcal{E} - \bar{\mathcal{E}}(\overline{E^0}, x)}{\sigma_u} - \frac{\Delta E^0}{\sigma_u}\frac{\partial \bar{\mathcal{E}}}{\partial E^0} + \cdots \qquad 24.14$$

Neglecting higher order terms in the expansion, and using 24.13:

$$\Delta_u = \frac{\Delta E^0}{\sigma_u}\left(1 - \frac{\partial \bar{\mathcal{E}}}{\partial E^0}\right) \qquad 24.15$$

Differentiating 24.10 with respect to E^0 and using 24.12:

$$0 = \frac{1}{\chi_1(\bar{E}^0)} - \left(1 - \frac{\partial \bar{\mathcal{E}}}{\partial E^0}\right)\frac{1}{\chi_1(E^f)} \qquad 24.16$$

Substituting in 24.15:

$$\Delta_u = \frac{\Delta E^0}{\sigma_u^0} \qquad 24.17$$

where σ_u^0 is given by (Eq.21.22):

$$\sigma_u^0 = \sigma_u \frac{\chi_1(\bar{E}^0)}{\chi_1(E^f)} = \chi_1(\bar{E}^0)\,\gamma^{-1/2}\left[f_2(\bar{E}^0) - f_2(E^f)\right]^{1/2} \qquad 24.18$$

$f_2(E^0)$ has been replaced by $f_2(\bar{E}^0)$ in accordance with the assumption that σ_u is essentially constant for all E^0 with which we are concerned. Using 24.8 and 24.17, we have:

$$P(E^0 - E^f, E^0; x) = \frac{1}{\sigma_u} P_{\lambda_u, \zeta_u, \dots}\left(\frac{\Delta E^0}{\sigma_u^0}\right)$$

$$\int_0^\infty P(E^0 - E^f, E^0; x)\, dE^0 = \int_{-\infty}^{\infty} \frac{\sigma_u^0}{\sigma_u} P_{\lambda_u, \zeta_u, \dots}(\Delta_u)\, d\Delta_u = \frac{\sigma_u^0}{\sigma_u} \qquad 24.19$$

and by 24.7:

$$S(E^0, E^f, x) = \frac{1}{\sigma_u^0} P_{\lambda_u, \zeta_u, \dots}\left(\frac{\Delta E^0}{\sigma_u^0}\right) \qquad 24.20$$

We have proved that, provided only that the spread in initial energies is sufficiently small, the initial energy distribution $S(E^0; E^f, x)$ is given by a curve with the same dimensionless parameters λ_u, ζ_u, ..., (same shape) as the distribution $P(\mathcal{E}, \bar{E}^0, x)$, and with a mean $\bar{E}^0$ and a fluctuation σ_u^0 given by 24.11 and 24.18.

For $E_h^f \to 0$, the equations giving the parameters for the initial energy distribution reduce to:

$$f_1(\overline{E_k^0}) = \gamma R$$

$$\sigma_u^0 = \chi_1(\overline{E_k^0})\,\gamma^{-1/2}\left[f_2(\overline{E_k^0})\right]^{1/2} \qquad 24.21$$

$$\lambda_u^0 = \lambda_u = \gamma^{1/2}\frac{f_3(\overline{E_k^0})}{\left[f_2(\overline{E_k^0})\right]^{3/2}}$$

The subscript "k" has been restored, and the distance x becomes the range R (in collision units). The formulas 24.21 have been derived using equations which are valid only when x is less than about 85% of the range (Sec.22). However, since the parameters depend only slightly on E_k^f for $E_k^f \ll \overline{E_k^0}$, the extrapolation to $E_k^f = 0$ is presumably justified.

The function λ_u^0 is the same as the skewness in range distribution given by 23.16 or Figure 10b. The fluctuation σ_u^0 is exactly the uncertainty in the initial energy as read from the (mean) range-energy relation if the range R is regarded as uncertain by an amount $\left(\overline{(\Delta R)^2}\right)^{1/2}$ as given by 23.15 or Figure 10a. The relation 24.21 for $\overline{E_k^0}$ is just the relation 23.15 for $\overline{R}$ with the bar changing places. The percentage fluctuation in initial energy $\sigma_u^0/\overline{E_k^0}$ is practically constant for $\overline{E_k^0}/\mu$ from 0.01 to 10; it is 2.7% at $\overline{E_k^0}/\mu = 0.01$, falls slowly to 2.1% at $\overline{E_k^0}/\mu = 1$, and rises to 3.3% at $\overline{E_k^0}/\mu = 10$; (these figures are for $\gamma = 1$). The connection between the distribution $Q(x,E)$ defined in Section 23 and the distribution $S(E^0, E^f, x)$ can obviously be treated in a similar way, using Bayes' theorem, to the treatment given above. This would enable us to derive the formulas 24.21 from

23.14, and thus justify the above extrapolation to $E_k^f = 0$.

25. Extensions and Other Applications.

The methods of Chapters III and IV are readily applicable to the determination of the distribution $P(\varepsilon, x)$ of energy loss by ionization for energies outside the range covered by the present calculations. The extension of the curves of Figures 5 - 10 to energies up to $E_k/\mu \sim 50$ is simply a matter of continuing the calculations up to this point. The corrections of order γw discussed in Section 22 become increasingly important at higher energies, and for $E_k/\mu > 50$, it would probably be necessary to include the terms $O(\gamma w)$ in the original computation. The computed curves would then apply only for a particular value of γ. For $E_k/\mu \gtrsim 50$, the correction terms due to spin (Section 6) become important; e.g. at $E_k/\mu = 50$, the correction to χ_3 is nearly 20% for spin ½. For $E_k/\mu \geq 20$, corrections due to spin should be applied to the results; the computation for $E_k/\mu \geq 50$ would probably have to be restricted to particles of a particular spin.

When E_m' is not large compared with I_z, collisions in which the resonance levels of the atom are excited contribute appreciably to the fluctuations as well as to the mean energy loss, even at large distances x. (For limitations at small distances x, see the discussion of Landau's results in Section 2.) In this case, the approximation made in the present calculations is no longer valid, of lumping the effect of such collisions together in a term X in the spectrum and neglecting the fluctuations produced. The general formulas of Chapters III and IV still apply, but the moments χ_2, χ_3, etc., must be computed for the correct collision spectrum including all types of collisions. The second moment can be calculated by using a sum rule, as sketched in Section 5, without determining the precise form of the spectrum. At low energies, $E'_m \doteq \frac{4\mu_e}{\mu} E_k$ (when terms $O(\frac{3\mu_e}{\mu})$ are neglected). The condition $E'_m \gg I_2$ becomes $E_k/\mu \gg \frac{I_2}{4\mu_e}$; for $I_2 \sim$ 1000 ev, $I_2/4\mu_e \sim 5 \times 10^{-4}$, this condition is well satisfied at the lower limit of energies considered in the present calculations ($E_k/\mu = 10^{-2}$). For lower energies, however, the collisions with $E' \sim I_2$ begin to contribute to χ_2.

For electrons at energies large compared with I_2, the Landau approximation for $P(\xi, x)$ is valid up to fairly large values of x. The collision spectrum $\chi(E')$ for electrons is given in Reference (4) (p. 244), and is practically equal to the spectrum 11.1 assumed by Landau for $E' < \frac{1}{4} E_k$. At $E' = \frac{1}{4} E_k$, the correct spectrum is about 10% larger than 11.1, and at $E' = \frac{1}{2} E_k$, it is 50% larger. Hence Landau's solution is valid for $I_2 \ll \xi \ll \frac{1}{4} E_k$, where "$\ll$" means smaller by a factor of perhaps 10. When these conditions are not satisfied, the more general methods of Chapter III may be applied to determine $P(\xi, x)$ for the correct spectrum $\chi(E')$. The condition $\xi \ll \frac{1}{4} E_k$ insures that collisions with $E' \gtrsim \frac{1}{4} E_k$ occur only rarely. Since these collisions are accompanied by fairly large angular deflections of the electrons, the treatment of the case when $\xi \ll \frac{1}{4} E_k$ is not satisfied requires a consideration of spatial scattering and of the geometry of the apparatus through which the electron is passing. When $\beta^2 < 1$, it is also required, either for Landau's solution or for the applicability of the more general methods of Chapter III, that ϵ_p (or $\bar{\epsilon}$) be small compared with E_k^0; otherwise the variation of $\chi(E')$ with E must be considered. The condition $\epsilon_p \ll E_k$ for electrons with $\beta^2 < 1$ is practically equivalent to the condition $\xi \ll \frac{1}{4}E_k$, so that the Landau approximation 11.1 to the spectrum, and the assumption that the spectrum is independent of E, both break down at about the same values of x. The methods of Chapter IV, (in combination perhaps with the method of Williams discussed in Section 17), would allow the solution to be extended somewhat further.

At high energies ($\sim 10^7$ - 10^8 ev for electrons, $\sim 10^{11}$ ev for mesons) energy loss by radiation is about as great as energy loss by collision. At lower energies, radiation is comparatively unimportant, its effect being to produce an occasional large energy loss. This effect can easily be taken into account by computing the probability for such a single large loss by radiation, and adding this to $P(\epsilon, x)$ as a very small tail extending to large values of $\epsilon - \epsilon_p$. The methods used in this thesis are not very well suited to treating the general problem of the distribution of energy loss by radiation, because such energy losses occur mainly by emission of large quanta which carry away an appreciable fraction of the kinetic energy at one time. The fluctuations are therefore quite large, and an initially homogeneous group of particles rapidly becomes spread over a wide range of energies. The method of Chapter IV of assuming a linear relation between the moments χ_1, χ_2, etc., and the deviation in energy loss ΔE would therefore be little better than neglecting the dependence on E altogether. The distribution curve $P(E,x)$ for radiation losses bears no resemblance to a Gaussian curve, so that expansions like those of Section 14 and 15 are of little use. An approximate solution for the distribution in energy loss due to radiation by high energy electrons has been given by Bethe and Heitler (35); this solution is based on an approximation to the radiation spectrum which underemphasizes the fluctuations, particularly at very high energies. A solution which somewhat overemphasizes the fluctuations has been given by Furry (38). In any case, the cascade multiplication which accompanies large radiative energy losses makes a discussion of energy loss alone of the primary particle rather academic.

The general problem of determining the distribution $P(E,x)$ of the total energy loss due to any sorts of processes which cause individual losses of energy E' whose probability per unit distance is given by a spectrum $\chi(E';E,x)$, leads to an equation of the form 9.7 or 9.8. When the dependence of the spectrum χ on E is neglected, the solution may be obtained by a Laplace transform as in Section 10. Most of the methods developed in Chapter III for evaluating this transform, (except those of Section 17), and the methods of Chapter IV for taking the dependence on E into account, are especially tailored to the case when the distribution $P(E,x)$ is roughly Gaussian, at least to the extent that it is unimodal (single humped) and confined more or less to a definite region of the energy scale for each value of x. The Edgeworth and weighted Edgeworth expansions are of little value unless the distribution is unimodal. The general formulas 12.16 and 12.17 for the moments are still applicable whenever χ is independent of E and may be used as suggested in Section 12 to determine the coefficients in other types of series which may be more suited to other kinds of distributions.

Sometimes the spectrum $\chi(E';E,x)$ is more conveniently expressed by replacing the variable E' by a variable $v = E'f(E)$; e.g. the probability of energy loss by radiation is most conveniently expressed in terms of the fractional energy loss $v = E'/E$ (4). The probability $\chi(v,E,x)$ for a collision with a particular value of v is then given. One way of treating this case is suggested in Cramèr's book (25,p.219,220). We study the distribution $P(z,x)$ of the sum $z = \sum v$, summed over all "collisions"; this distribution is determined by an equation similar to 9.7 or 9.8. We then observe that if each E' is a fairly small fraction of E, we can relate the variable z to E by:

$$z \doteq \int_{E^0-E}^{E^0} f(E)\,dE \qquad 25.1$$

Integrals often arise in mathematical physics of the form:

$$F(\mathcal{E},x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\mathcal{E} + xf(s)}\,ds \qquad 25.2$$

The function $F(\mathcal{E},x)$ will frequently be a distribution function of some sort, but in any case the analogy between 25.2 and 10.10 will often allow us to regard $F(\mathcal{E},x)$ as a distribution in energy loss resulting from a certain energy loss spectrum $\chi(E')$ whose transform is:

$$\tilde{\chi}(s) = f(s) - f(c \pm i\infty) \qquad 25.3$$

If $f(0) - f(c \pm i\infty)$, which we interpret as the mean number of "collisions" per unit "distance" x, is infinite, we may be able to remove the infinite part as a term $-s\mathcal{I}$, or we may define $\chi(E')$ by:

$$f(s) - f(0) = \int_0^\infty (e^{-sE'} - 1)\,\chi(E')\,dE' \qquad 25.4$$

It is really only the integral 25.4, and not the Laplace integral $\tilde{\chi}(s)$, which must converge for a mathematically allowable collision spectrum. The function $\chi(E')$ is often known, or can more easily be found than the function 25.2. If $f(0) \neq 0$ then the function $F(\mathcal{E},x)$ is not normalized, and we would interpret the energy loss distribution as the normalized function:

$$e^{-xf(0)}F(\mathcal{E},x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\mathcal{E} + x[\tilde{\chi}(s) - \tilde{\chi}(0) - s\mathcal{I}]}\,ds \qquad 25.5$$

This interpretation of the equation 25.2 allows us to make use of the sort of arguments described in Section 17, in evaluating the integral. The series expansion methods of Chapter III are always available for the evaluation of integrals of the type 25.2, the weighted and unweighted Edgeworth expansions being useful when $F(\mathcal{E},x)$ is unimodal in $\mathcal{E}$. These remarks can also be extended to the more general case when a function $f(s,x)$ occurs in 25.2.

26. Instructions for Using the Curves.

The curves presented in Figures 2 - 9 allow the determination of the distribution in energy loss by ionization, of mesons of energies from 1 to 1000 Mev, in traversing thicknesses of matter up to 85% of the range of the particles; for greater thicknesses, the distribution in range, determined by Figure 10, will usually be of more interest. The curves can be used for protons of energies from 10 to 10,000 Mev. The probability distribution of the initial energy of a particle which has traversed a known thickness and emerged with a known final energy, (which may be zero), can also be found from the curves, as will be explained below. For the convenience of the reader, the definitions of the symbols to be used in this section are repeated here:

$P(\mathcal{E})d\mathcal{E}$ = Probability of a total energy loss between $\mathcal{E}$ and $\mathcal{E}+d\mathcal{E}$

Weighted Parameters
$\mathcal{E}_p$ = most probable energy loss
σ_w = weighted r.m.s. fluctuation
λ_w = weighted skewness

Unweighted Parameters

$\bar{\mathcal{E}}$ = mean energy loss

σ_u = (unweighted) r.m.s. fluctuation $= \left(\overline{(\Delta\mathcal{E})^2}\right)^{1/2}$

λ_u = (unweighted) skewness $= \overline{(\Delta\mathcal{E})^3} / \left(\overline{(\Delta\mathcal{E})^2}\right)^{3/2}$

E_k^0 = Initial kinetic energy

$\bar{E}_k$ = final mean kinetic energy $= E_k^0 - \bar{\mathcal{E}}$

μ = rest energy of ionizing particle

μ_e = rest energy of electron ($2\mu_e \doteq 1$ Mev)

$\gamma = \frac{200\mu_e}{\mu}$ ($\gamma = 1$ for meson)

t = thickness of meterial traversed in cm.

x = thickness of material traversed in "collision units": $x = 0.150 \frac{Z}{A} \rho t$

$\frac{Z}{A}$ = ratio of atomic number to atomic weight for material traversed. (If material is not monatomic, use $\Sigma Z / \Sigma A$.)

ρ = density of material in gm/cm^3.

I_z = average ionization potential of all the electrons. To a sufficiently good approximation, $I_z \doteq 13.5 Z$ ev.

w = fractional total energy $= \frac{E_k}{\mu} + 1$

$\beta^2 = v^2/c^2 = 1 - \frac{1}{w^2} = 2\frac{E_k}{\mu}\left[1 - \frac{E_k^2}{\mu^2}\left(\frac{2w-1}{w}\right)\right]$

χ_1 = mean rate of energy loss per collision unit. (Fig. 5a)

q = ionization potential conversion factor. (Fig.5b).

R = range

f_1, f_2, f_3, g_3 = functions of E_k/μ plotted in Figures 6,7, 8,9.

$\xi, 1/c$ = (see Eq.26.2 below)

The thickness t of material traversed always occurs in the combination $(0.150\, Z/A\, \rho)\, t$, and it is convenient to define the "collision thickness" x as above. When thicknesses are measured in terms of the dimensionless quantity x, most of the variations in ionization losses in different kinds of material are eliminated. The thicknesses of various materials at which $x = 1$ are given in Table 9. $\bar{E}$, σ_u, and λ_u are the ordinary mean energy loss, r.m.s. fluctuation, and skewness. The weighted parameters E_p, σ_w, λ_w, correspond to these same quantities calculated after multiplying the distribution $P(E)$ by a weight factor which reduces the effect of the tail of the curve. In the limiting case of small thicknesses, λ_u and $\sigma_u/\bar{E}$ become infinite, whereas λ_w and σ_w/E_p remain finite. For large thicknesses, E_p, σ_w, and λ_w approach $\bar{E}$, σ_u, and λ_u.

The distribution curves are most conveniently plotted in terms of the weighted parameters as in Figure 2. The energy loss is measured in terms of the variable $(E-E_p)/\sigma_w$; i.e. 0 on the horizontal scale corresponds to an energy loss E_p, and energy is measured along the horizontal scale in units of σ_w. We then have a family of curves characterized by the single parameter λ_w, which varies in value from 1.48 (for very small thicknesses) to 0 (Gaussian curve). In order to use the curves of Figure 2, the three weighted parameters E_p, σ_w, λ_w, must be determined in accordance with the instructions given below. The differential distribution curves $\frac{\sigma_w}{F} P(E)$ as plotted in Figure 2 are not normalized. To normalize in the scale of the variable $(E-E_p)/\sigma_w$, multiply by the factor F, which depends on λ_w and is also plotted in Figure 2; the curves then all have the same area. To normalize in the scale of the variable $\mathcal{E}$, (probability per unit energy), multiply by F/σ_w; this gives just $P(\mathcal{E})$ as defined above. The integral distribution curves $\int_{\mathcal{E}}^{\infty} P(\mathcal{E})d\mathcal{E}$ as plotted in Figure 2, give directly the probability that a particle will lose an energy greater than $\mathcal{E}$. The distribution curves of Figure 2 are accurate to within about plus or minus one quarter of the distance between them, when used with the proper parameters as determined below. The integral curves for $\lambda_w \geq 1.45$ are most readily obtained from the formulas:

$$\int_{\mathcal{E}}^{\infty} P(\mathcal{E})d\mathcal{E} \doteq \begin{cases} 0 & \left(\frac{\mathcal{E}-\mathcal{E}_p}{\xi} > c\right) \\ \frac{\xi}{\mathcal{E}-\mathcal{E}_p} - 1/c - \beta^2/c \log \frac{c\xi}{\mathcal{E}-\mathcal{E}_p} & \left(10 < \frac{\mathcal{E}-\mathcal{E}_p}{\xi} < c\right) \\ 1 - \frac{F}{0.663} + \frac{F}{0.663}\left[\int_{\mathcal{E}}^{\infty} P(\mathcal{E})d\mathcal{E}\right]_{\lambda_w = 1.48} & \left(\frac{\mathcal{E}-\mathcal{E}_p}{\xi} < 10\right) \end{cases} \qquad 26.1$$

where:

$$F = 0.663\left[1 + \frac{1}{c} + \frac{\beta^2}{c} \log c/10\right] \qquad (\lambda_w \geq 1.45)$$

and ξ and c are given by formulas 26.2 below.

For small thicknesses, Figure 3 may be used to determine the weighted parameters $\mathcal{E}_p$, σ_w, λ_w directly. Small thicknesses are defined as those for which $\bar{\mathcal{E}} \lesssim \frac{\mathcal{E}_k^0}{10}$, or $\lambda_w \gtrsim 1$, whichever condition is least restrictive. (The former condition applies in general for energies up to $\frac{\mathcal{E}_k^0}{\mu} \sim 1$, and the latter at higher energies.) The curves of Figure 3 are self-explanatory. We first calculate the quantities:

$$\xi = \frac{2\mu_e}{\beta^2}\left(0.150\frac{Z}{A}\right)\rho t = \frac{2\mu_e k}{\beta^2} \qquad [\text{energy}]$$

26.2

$$1/c = \frac{1+\frac{2\mu_e}{\mu}w}{w^2-1}\,\frac{\xi}{2\mu_e}$$ [dimensionless]

The dimensionless quantity $1/c$ is proportional to the thickness t, and the curves of Figure 3 give, as functions of $1/c$, the weighted skewness λ_w, and two other numbers b and j related to σ_w and $\mathcal{E}_p$ by the formulas:

$$\sigma_w = b\xi$$

$$\mathcal{E}_p = \xi\left(\log\frac{\xi}{\epsilon}+j\right) \qquad 26.3$$

where:

$$\epsilon = \frac{I_z^2}{2\mu_e(w^2-1)}e^{\beta^2}$$

The curves of Figure 3 and the formulas 26.2 and 26.3 may be used (with Figure 2) for meson energies from 1 to 50,000 Mev. Above 50,000 Mev, the formulas are correct only for mesons of spin 0; the effects of spin become important at about 50,000 Mev. The curves are not valid for extremely small thicknesses when $\xi \lesssim 5I_z$, but such a case is not likely to occur; (it corresponds, for example, to about 0.005 cm of lead or less).

For large thicknesses ($\bar{\mathcal{E}} > \frac{E_k^0}{10}$ or $\lambda_w < 1$), the unweighted parameters $\bar{\mathcal{E}}$, σ_u, λ_u, should first be determined. (This method may be used for any thickness provided the numbers required can be read from the curves with sufficient accuracy. Figure 3, on the other hand, should not be used except when the condition for small thickness is satisfied.) The mean final energy is determined from Figure 6, using the formula:

$$f_1(\bar{E}_k) = f_1(E_k^0) - \gamma x \qquad 26.4$$

For materials other than air, iron, or lead, (or with an equivalent I_z), interpolate geometrically, according to I_z, between the curves in Figure 6, or multiply $f_1(E_k)$ by the correction factor $q(\frac{E_k}{2})$ given in Figure 5b. σ_u is then computed from Figures 5 and 7 and the formula:

$$\sigma_u = \chi_1(\bar{E}_k)\gamma^{-1/2}[f_2(E_k^0) - f_2(\bar{E}_k)]^{1/2} \qquad 26.5$$

When I_z is not approximately 330 ev. (iron), f_2 and χ_1 as plotted in Figures 5 and 7 must be corrected as indicated in the figures. $f_2(E_k)$ should be multiplied by $[q(E_k)]^3$ (Figure 5b), and $\chi_1(E_k)$ should be divided by $q(E_k)$. λ_u is computed from Figures 7,8, and 9 and the formula:

$$\lambda_u = \gamma^{1/2}\left\{\frac{f_3(E_k^0) - f_3(\bar{E}_k)}{[f_2(E_k^0) - f_2(\bar{E}_k)]^{1/2}} + \frac{g_3(\bar{E}_k)}{[f_2(E_k^0) - f_2(\bar{E}_k)]^{1/2}}\right\} \qquad 26.6$$

A correction must also be applied here when $I_z \neq 330$ ev. The most convenient procedure is to compute λ_u from 26.6, using the values read from Figures 7,8, and 9, and then multiply by $[q(E_k^0)]^{-1/2}$. The last term in 26.6 is a small correction, negligible for $\bar{E}_k \lesssim 0.1\mu$ and amounting to about 20% at $\bar{E}_k = 7\mu$. A small correction factor should be applied to 26.5 and 26.6 when $w \sim 10$; σ_u should be multiplied by $(1-0.004\gamma w)$, and λ_u by $(1-0.005\gamma w)$. When $\bar{\mathcal{E}}\ (= E_k^0 - \bar{E}_k)$, σ_u, and λ_u are determined, the weighted parameters are readily determined from Figure 4. λ_w is determined first from the curve λ_u vs λ_w; σ_w is then determined from the curve σ_w/σ_u vs λ_w; and finally $\mathcal{E}_p$ from the curve $(\mathcal{E}-\mathcal{E}_p)/\sigma_w$ vs λ_w. For $\lambda_u \lesssim 1$, (as it is for large thicknesses), it is usually a good enough approximation to take $\lambda_w \doteq \lambda_u$.

The curves in Figure 10 giving the fluctuation and skewness in the range distribution vs the initial energy are self-explanatory. A correction is required when $I_z \neq$ 330 ev, as indicated in Figure 10; for this purpose, q should be evaluated at the initial energy E_k^0. The skewness in range $\overline{(\Delta R)^3} / (\overline{(\Delta R)^2})^{3/2}$ is of the order of -0.3 to -0.5 (for mesons), so the distribution curve for the range will have the shape shown in Figure 2 for $\lambda_w =$ 0.3 to 0.5, except that the curve is now reversed (negative λ) so that the larger tail is on the side of smaller ranges. In using Figure 2 as a range distribution curve, one should of course take $\sigma_u = (\overline{(\Delta R)^2})^{1/2}$. The mean range $\bar{R}$ can be determined from Figure 6, using the formula:

$$\bar{R} = \frac{1}{\gamma} f_1(E_h^0) \qquad \text{(collision units)}$$

$$= \left[0.150 \frac{Z}{A} \rho\right]^{-1} \frac{1}{\gamma} f_1(E_h^0) , \qquad \text{(cm)} \qquad 26.7$$

It should be noted that $\bar{R}$ is not the peak of the range distribution, but is slightly displaced toward smaller ranges, due to the somewhat larger tail on the small range side. The difference $(R_p - \bar{R}) / (\overline{(\Delta R)^2})^{1/2}$, (R_p is the most probable range), will be given by the curve $(\bar{E} - E_p)/\sigma_w$ in figure 4, so that we have ($\lambda \sim 0.4$, $q = 1$):

$$\frac{R_p - \bar{R}}{(\overline{(\Delta R)^2})^{1/2}} \doteq 0.2 \qquad \text{(mesons)}$$

$$\frac{R_p - \bar{R}}{\bar{R}} \doteq 0.8\% \qquad \text{(mesons)} \qquad 26.8$$

A displacement from the peak of 0.2 on the horizontal scale of Figure 2 would be just perceptible.

If the distance x traversed and the final energy E_k^f are known, the mean initial energy $\bar{E}_k^0$, and the fluctuation in the initial energy are given by:

$$f_1(\overline{E_k^0}) = f_1(E_k^f) + \gamma x$$

$$\sigma_u^0 = \left(\overline{(\Delta E_k^0)^2}\right)^{1/2} = \chi_1(\overline{E_k^0})\gamma^{-1/2}\left[f_2(\overline{E_k^0}) - f_2(E_k^f)\right]^{1/2} \qquad 26.9$$

The skewness λ_u^0 in the initial energy distribution is given by the formula 26.6 with E_k^0 and $\overline{E_k}$ replaced by $\overline{E_k^0}$ and E_k^f, respectively. Corrections for $I_z \neq 330$ ev are the same as explained above for equations 26.4, 26.5, 26.6. Thus the energy loss distribution when the final energy is known is essentially the same as when the initial energy is known, except that σ_u^0 has the factor $\chi_1(\overline{E_k^0})$ instead of $\chi_1(\overline{E_k^f})$, which may make a considerable difference in the spread of the curve. When $E_k^f = 0$, i.e. when the range R is known, the formulas 26.8 reduce to:

$$f_1(\overline{E_k^0}) = \gamma R \qquad (R \text{ in collision units})$$

$$\sigma_u^0 = \chi_1(\overline{E_k^0})\gamma^{-1/2}\left[f_2(\overline{E_k^0})\right]^{1/2} \qquad 26.10$$

$$\lambda_u^0 = \gamma^{1/2}\frac{f_3(\overline{E_k^0})}{\left[f_2(\overline{E_k^0})\right]^{3/2}} \qquad (\sim 0.3 \text{ to } 0.5 \text{ for mesons})$$

The ratio $\sigma_u^0/\overline{E_k^0}$, when R is known, turns out to be about 2.4% ±0.3% for mesons of (initially) 1 to 500 Mev, and is 3.3% at 1000 Mev ($q=1$).

As an example of a "small thickness" calculation, we consider a meson of 500 Mev, traversing 3.8 cm of lead. Using Table 9, we find:

$$x = \frac{3.8}{1.48} = 2.57$$

We now compute:

$$w = \frac{E_k}{\mu} + 1 = \frac{500}{100} + 1 = 6$$

$$\beta^2 = 1 - \frac{1}{w^2} = 0.972$$

$$\xi = \frac{2\mu_e x}{\beta^2} = \frac{2.57}{0.972}(2\mu_e) \doteq 2.65 \text{ Mev (setting } 2\mu_e \doteq 1 \text{ Mev)}$$

$$1/c = \frac{1+0.01w}{w^2-1} \; \frac{\xi}{2\mu_e} = 0.0803$$

From Figure 3, we read the values:

$$\lambda_w = 1.3$$

$$b = 1.35$$

$$j = 0.18$$

Using $I_z \doteq 13.5Z$ = 1000 ev, we now compute:

$$\sigma_w = b\xi = 3.58 \text{ Mev}$$

$$\epsilon = \frac{I_z^{\,2}}{2\mu_e(w^2-1)} e^{\beta^2} = 2\mu_e e^{\beta^2} \frac{\left(\frac{I_z}{2\mu_e}\right)^2}{w^2-1} = e^{\beta^2} \frac{10^{-6}}{35} \quad \text{Mev}$$

$$\mathcal{E}_p = \xi\left(\log \xi/\epsilon + j\right) = 2.65\left[\log(2.65 \times 35 \times 10^{9} - \beta^2 - j\right] \quad \text{Mev}$$

$$= 46.5 \text{ Mev}$$

The distribution curve is therefore midway between the curves λ_w= 1.2 and λ_w= 1.4 of Figure 2, with $(\mathcal{E}-\mathcal{E}_p)/\sigma_w = 0$ corresponding to a loss of 46.5 Mev, and with each unit on the horizontal scale representing 3.58 Mev. The mean energy loss, according to Figure 4, occurs at $(\bar{\mathcal{E}}-\mathcal{E}_p)/\sigma_w = 1.05$, or $(\bar{\mathcal{E}}-\mathcal{E}_p) = 3.8$ Mev. Therefore the mean energy loss is $\bar{\mathcal{E}} = 50.3$ Mev. A check on the work is afforded by noting that $\bar{\mathcal{E}} \doteq \mathcal{X}_1 x$ (small thicknesses). From Figure 5a and 5b:

$$\mathcal{X}_1 = (2\mu_e)\,\frac{21.5}{1.115} = 19.3 \qquad \text{Mev/collision unit}$$

$$\bar{\mathcal{E}} = \mathcal{X}_1 x = 2.57 \times 19.3 = 49.5 \qquad \text{Mev}$$

This agrees with the value of $\bar{\mathcal{E}}$ obtained above, within the accuracy with which the numbers can be read from the curves.

As an example of a "large thickness" calculation, we consider a meson with initial energy 20 Mev ($E_k^0/\mu = 0.2$) traversing 1.5 gm/cm^2 of silver chloride. Using Table 9, we find for x:

$$x = \frac{1.5}{14.9} = 0.101$$

For Ag, $I_z \doteq 13.5 \times 47 = 630$ ev. For Cl, $I_z \doteq 13.5 \times 17 = 230$ ev. For the average I_z for the 47 plus 17 electrons, we have:

$$I_z = \frac{(47 \times 630) + (17 \times 230)}{47+17} = 520 \text{ ev}$$

A glance at Figure 5b shows that silver chloride is to be interpolated about 1/3 of the way from the iron to the lead curve. Using this rule, we find from Figure 6 and equation 26.4:

$$f_1(\bar{E}_k) = f_1(E_k^0) - \gamma x = 0.277 - 0.101 = 0.176$$

At $f_1 = 0.176$, we find $\bar{E}_k/\mu = 0.154$, or $\bar{E}_k = 15.4$ Mev. The mean energy loss is $\bar{\varepsilon} = 4.6$ Mev. Using equations 26.5, 26.6, and Figures 5, 7, 8, 9, we find:

$$\sigma_u = \chi_1(\bar{E}_k)\gamma^{-1/2}\left[f_2(E_k^0) - f_2(\bar{E}_k)\right]^{1/2} = 54\left[(8.8-3.7)\times 10^{-5}\right]^{1/2} = 0.386 \text{ Mev}$$

$$\lambda_u = \gamma^{1/2}\left\{\frac{f_3(E_k^0) - f_3(\bar{E}_k)}{\left[f_2(E_k^0) - f_2(\bar{E}_k)\right]^{3/2}} + \frac{g_3(\bar{E}_k)}{\left[f_2(E_k^0) - f_2(\bar{E}_k)\right]^{1/2}}\right\}$$

$$= \frac{(2.6-0.74)\times 10^{-7}}{(51)^{3/2}\times 10^{-9}} + \frac{7.8\times 10^{-5}}{(51)^{1/2}\times 10^{-3}}$$

$$= 0.51 + 0.011 = 0.52$$

(The g_3 term could have been neglected.) These values are for $I_z = 330$ ev. To correct for $I_z = 520$ ev, we multiply σ_u by $q^{1/2}$, and λ_u by $q^{-1/2}$. From Figure 5b:

$$q^{1/2} = 1.03$$

(When $\bar{E}_k \ll E_k^0$, σ_u should be multiplied by $[q(E_k^0)]^{3/2}/q(\bar{E}_k)$, λ_u by $[q(E_k^0)]^{-1/2}$.) The corrected values are:

$$\sigma_u = 0.398 \text{ Mev.}$$

$$\lambda_u = 0.49$$

From Figure 4, we find ($\beta^2 = 0.4$):

$$\lambda_w = 0.5$$

$$\sigma_w = 0.93 \times 0.398 = 0.37 \text{ Mev}$$

$$\bar{\varepsilon} - \varepsilon_p = 0.25 \times 0.37 = 0.09 \text{ Mev}$$

$$\varepsilon_p = 4.5 \text{ Mev}$$

Thus the curve $\lambda_w = 0.5$ in Figure 2 represents the energy loss distribution, with the peak of the differential curve representing a loss of 4.5 Mev, and each unit on the horizontal scale representing 0.4 Mev.

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K. R. Symon

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Table 1

System of Units

Quantity	Unit	Symbol
Fundamental Units:		
Length	Centimeter	cm
Velocity	velocity of light	c
Energy	electron-volt	ev
Derived or Special Units:		
Mass	electron-volt/c^2	ev/c^2
Density of braking material	gram per cubic centimeter	gm/cm^3
Path length in braking material	collision unit	$\left[\pi r_0^2 N \frac{Z}{A} \rho\right]^{-1}$ $= \left[0.150 \frac{Z}{A} \rho\right]^{-1}$

Table 2

List of Important Symbols Used

(Latin letter symbols are listed first in alphabetical order, followed by Greek letter symbols.)

Symbol	Meaning	First Important Use (Equation or Section No.)
A	constant characterizing braking material [energy/cm]	2.4
A	coefficient defined by	15.14
A	Atomic weight of braking material	2.5
$A(E^0)$	a priori initial energy distribution	24.3
A_n	n^{th} coefficient in Gram-Charlier series	14.1
$\underline{A_n}$	$A_n/\sigma_u{}^n$	14.7
a	variable introduced by	15.12
a^+	a as given by the "$\beta^2=0$" formulas	16.22
B	coefficient defined by	15.14
$B_{k_1\cdots k_n}$	number defined by	12.18
b	variable introduced by	15.12
b^+	b as given by the "$\beta^2=0$" formula	16.12
C	Euler's constant: 0.577	11.3
c	velocity of light	
c	real part of limit of integration	Theorem III, Sec. 10

Symbol	Meaning	First Used
c	E'_m/ξ	14.21
d	variable introduced by	15.12
$d\sigma$	differential collision cross section	4.1
E	kinetic plus rest energy (incident particle)	4.1
E_c	characteristic energy depending on μ	4.5
E_k	kinetic energy of incident particle	4.6
E_n	excitation energy of n^{th} atomic state	5.4
E_k^f	final kinetic energy	23.14
E^0	initial total energy	21.11
E_k^0	initial kinetic energy	21.11
E'	energy lost in a single collision	Section 1
E'_m	maximum possible value of E'	4.1
E'_T	a specially chosen value of E'	2.2
ε	total energy lost	Section 1
$\bar{\varepsilon}$	mean total energy lost	2.1
ε_p	most probable energy loss	2.3
$\overline{\varepsilon^n}$	n^{th} moment of $P(\varepsilon,x)$	12.16
$Ei(x)$	exponential integral	16.3

Symbol	Meaning	First Used
e	charge of electron	
F	normalizing factor	Section 18
$f_1, f_2, f_3,$ f_{3a}, f_{3b}	functions of E_k (or E) defined by	21.13 - 21.18
g	function defined by	16.18
g_3	function of E_k (or E) defined by	21.17
H_n	n^{th} Hermite polynomial	14.9
h	function defined by	16.18
I_z	mean ionization potential of atomic electrons	2.6
j	a numerical constant	2.4
j	a numerical function	16.10
K	wave number	5.1
$\vec{K}$	wave vector	5.1
$K(E)$	function defined by	7.2
k	function defined by	16.18
m	mass of electron (gm)	4.1
N	Avogadro's number	2.5
$P(\varepsilon,t)$	energy loss distribution	Section 1
$P(\varepsilon,x)$	energy loss distribution	9.1

Symbol	Meaning	First Used
$P_L(\epsilon,x)$	distribution given by Landau approximation	17.11
$P_w(\epsilon,x)$	weighted distribution function	15.22
$P_{\lambda_w}(\Delta_w)$	distribution with skewness λ_w normalized in scale of Δ_w	18.1
$\bar{P}(s,x)$	Laplace transform of $P(\epsilon,x)$	10.3
p	linear momentum	5.1
$Q(x,\epsilon)$	distribution in path length	23.1
q	ionization potential conversion factor	21.28
R	range	21.20
R	Rydberg constant	5.5
$\bar{R}$	mean range	23.15
r_0	classical radius of electron	4.1
$S(\epsilon^0,\epsilon^f,x)$	initial energy distribution	24.1
s	variable introduced by Laplace transformation	10.1
s_0	saddle-point in the s-plane	15.1
t	thickness of material traversed in cm	Section 1
v	velocity in cm/sec	
υ	fractional energy loss (by radiation)	7.1

Symbol	Meaning	First Used
w	E/μ	14.19
w_k	E_k/μ	14.20
X	rate of continuous energy loss	9.5
x	path length in collision units: $[0.150\frac{Z}{A}\rho)t$	Section 9
$\bar{x}$	mean path length	23.8, 23.14
Z	atomic number of braking material	2.5
β	v/c (incident particle)	4.1
γ	$200\mu_e/\mu$	Section 9
γ	Euler's constant: e^C	16.6
Δ_L	$(\epsilon-\epsilon_p)/\xi$	11.5
Δ_u	$(\epsilon-\bar{\epsilon})/\sigma_u$	14.4
Δ_w	$(\epsilon-\epsilon_p)/\sigma_w$	15.10
Δ'	variable introduced by	15.18
$\overline{(\Delta\epsilon)^n}$	n^{th} central moment of $P(\epsilon,x)$	12.17
$\overline{(\Delta R)^n}$	n^{th} central moment of range distribution	23.15
$\overline{(\Delta x)^n}$	n^{th} central moment of $Q(x,\epsilon)$	23.10
ϵ	constant characterizing braking material [energy]	2.4

Symbol	Meaning	First Used
ϵ_{0n}	matrix element	5.1
ζ_u	unweighted coefficient of excess $(=\chi_4/\sigma_u^4)$	16.20
ζ_w	weighted coefficient of excess	15.10
ζ^+	ζ_w as given by the "$\beta^2=0$" formula	16.14
η	an energy satisfying: $I_z \ll \eta \ll E'_m$	6.7
θ	variable of integration	14.5
χ_ν	νth semi-invariant	12.8
λ, λ_u	unweighted skewness $(=\lambda_3)$	14.24,ff.
λ_n	nth dimensionless moment of $P(\epsilon,x)$	12.24
λ_w	weighted skewness	15.10
λ^+	λ_w as given by the "$\beta^2=0$" formula	16.13
μ	rest energy of incident particle	4.1
μ_e	rest energy of electron	2.5
ξ	an energy proportional to t	2.3
$\Pi(\omega)$	Fourier transform of $P(\epsilon)$	12.5
ρ	density of braking material in gm/cm^3	2.5
σ_n	cross section for excitation of nth atomic state	5.1
σ_u	unweighted r.m.s. fluctuation	12.27

Symbol	Meaning	First Used
σ_w	weighted r.m.s. fluctuation	15.10
Φ	integral of Gaussian error function	14.13
$\phi(\Delta_L)$	Landau distribution function	2.3
φ	Gaussian error function	14.8
$\phi(v)$	radiation spectrum	7.2
χ	constant in Bloch energy loss formula	5.6
$\chi(E')$	collision spectrum	Section 1
$\chi(E',E)$	collision spectrum dependent on E	6.1
$\chi(E',E,x)$	collision spectrum as function of E and x	Section 9
$\chi(E',\mathcal{E},x)$	collision spectrum as function of $\mathcal{E}$ and x (any or all of the arguments may sometimes be omitted)	9.1
χ_n	nth moment of collision spectrum	12.1
$\chi_n(s)$	generalized nth moment	15.3
$\chi_n^+(s)$	$\chi_n(s)$ as given by the "$\beta^2=0$" formula	16.4
$\underline{\chi_n}$	nth dimensionless moment	12.25
$\underline{\chi_n}(s)$	generalization of $\underline{\chi_n}$	15.8
$\tilde{\chi}(s)$	Laplace transform of $\chi(E')$	10.3
ψ	wave function	5.1
ω	Fourier transform variable	12.4

Table 3

Weighted Parameters and Related Quantities

Computed for $\beta^2 = 0$

c/a	g	h	k	a	c	b	λ_w	j
10.0	1.000	1.000	0.998	2.178	21.78	1.477	1.477	0.201
5.0	0.997	0.969	0.945	2.175	10.88	1.470	1.429	0.199
3.0	0.975	0.864	0.857	2.28	6.84	1.473	1.305	0.234
2.0	0.930	0.740	0.794	2.48	4.96	1.465	1.166	0.282
1.5	0.882	0.645	0.760	2.68	4.02	1.443	1.056	0.309
1.0	0.795	0.525	0.728	2.96	2.96	1.368	0.903	0.289
0.85	0.757	0.483	0.723	3.07	2.61	1.327	0.847	0.261
0.7	0.709	0.434	0.706	3.20	2.24	1.268	0.777	0.212
0.6	0.672	0.399	0.703	3.32	1.992	1.224	0.727	0.169
0.5	0.627	0.365	0.695	3.38	1.690	1.153	0.671	0.081
0.3	0.509	0.281	0.684	3.58	1.075	0.964	0.532	0.208
0.2	0.425	0.225	0.678	3.75	0.750	0.824	0.436	0.478

$g \doteq (c/a)^{1/2}\left(1 - \frac{1}{4} c/a\right) \qquad (c/a < 0.2)$

$h \doteq \frac{1}{2}(c/a)^{1/2}\left(1 + \frac{1}{12} c/a\right) \qquad (c/a < 0.2)$

$k \doteq \frac{2}{3}\left(1 + \frac{1}{12} c/a\right) \qquad (c/a < 0.6)$

$a \doteq 4 - 0.351\, c \qquad (c/a \leq 1.0)$

$c \doteq \frac{96}{7} - \frac{24}{7} a \qquad (c/a < 0.2)$

$b \doteq c^{1/2}\left(1 - \frac{c}{16}\right) \qquad (c/a < 0.2)$

$\lambda_w \doteq \frac{1}{2} c^{1/2}\left(1 + c/48\right) \qquad (c/a < 0.2)$

$j \doteq \log c - c/4 \qquad (c/a < 0.2)$

Table 4

Weighted Parameters

Computed for $\beta^2 = 0.4, 0.7, 1$

		$\overset{+}{a}/a$			a	
c/a	β^2: 0.4	0.7	1	0.4	0.7	1
0	1.15	1.26	1.333	3.49	3.18	3.00
0.0251	1.15	1.26	1.325	3.47	3.16	3.00
0.1028	1.14	1.24	1.308	3.41	3.14	2.98
0.275	1.13	1.23	1.28	3.19	2.94	2.83
0.433	1.12	1.21	1.25	3.10	2.86	2.77
0.608	1.11	1.19	1.22	2.96	2.77	2.70
1.013	1.09	1.15	1.16	2.72	2.57	2.55
2.015	1.05	1.07	1.05	2.36	2.32	2.36
3.09	1.025	1.025	0.998	2.22	2.22	2.28
4.59	1.005	0.998	0.976	2.17	2.18	2.23
5.51	1.000	0.995	0.976	2.18	2.19	2.23
9.19	0.999	0.993	0.985	2.18	2.20	2.21

		c			b	
c/a	β^2: 0.4	0.7	1	0.4	0.7	1
0	0	0	0	0	0	0
0.0251	0.087	0.079	0.075	0.26	0.22	0.19
0.1028	0.351	0.323	0.307	0.54	0.44	0.37
0.275	0.882	0.813	0.781	0.794	0.691	0.600
0.433	1.34	1.24	1.20	0.946	0.829	0.729
0.608	1.80	1.68	1.64	1.053	0.930	0.826
1.013	2.75	2.61	2.59	1.198	1.076	0.973
2.015	4.76	4.68	4.76	1.328	1.235	1.158
3.09	6.84	6.84	7.01	1.376	1.310	1.256
4.59	9.95	10.02	10.25	1.410	1.366	1.332
5.51	12.0	12.06	12.3	1.422	1.385	1.352
9.19	20.0	20.1	20.3	1.442	1.425	1.404
18.36	40.0	40.0	40.0	1.461	1.450	1.444
45.9	100.	100.	100.	1.475	1.468	1.462

	λ_w			j		
c/a β^2:	0.4	0.7	1	0.4	0.7	1
0	0	0	0	$-\infty$	$-\infty$	$-\infty$
0.0251	0.15	0.14	0.13	-2.83	-3.20	-3.60
0.1028	0.31	0.28	0.27	-1.42	-1.88	-2.19
0.275	0.489	0.462	0.417	-0.68	-1.02	-1.30
0.433	0.597	0.566	0.512	-0.36	-0.67	-0.93
0.608	0.694	0.658	0.599	-0.18	-0.45	-0.68
1.013	0.868	0.822	0.749	0.04	-0.18	-0.36
2.015	1.111	1.052	0.973	0.20	0.07	-0.06
3.09	1.251	1.192	1.115	0.25	0.15	0.06
4.59	1.353	1.301	1.240	0.28	0.22	0.15
5.51	1.394	1.349	1.296	0.29	0.24	0.18
9.19	1.443	1.414	1.383	0.33	0.29	0.26
18.36	1.473	1.448	1.435	0.34	0.32	0.30
45.9			1.46	0.37	0.36	0.36

Table 5

Energy Loss Distribution Functions

Differential Distribution $\frac{\sigma_w}{F} P(\epsilon)$:

A_w	λ_w: 1.48	1.4	1.2	1.0	0.8	0.5	0
-3.0							.004
-2.5	.001					.003	.018
-2.0	.007					.025	.054
-1.5	.052					.093	.130
-1.0	.218					.230	.242
-0.5	.366						.352
0.0	.399					.399	.399
0.5	.357				(.367)	.357	.352
1.0	.283				.278	.270	.242
1.5	.214				.201	.172	.130
2.0	.167			.158	.128	.095	.054
2.5	.130			.107	.077	.048	.018
3.0	.105		.098	.071	.041	.024	.004
3.5	.084		.071	.044	.024	.011	.001
4.0	.067	.065	.048	.026	.015	.005	
4.5	.055	.053	.032	.016	.009	.002	
5.0	.047	.043	.022	.010	.005		
5.5	.040	.035	.015	.007	.002		
6.0	.033	.028	.010	.004	.001		
6.5	.029	.022	.007	.003			
7.0	.025	.018	.004	.002			
7.5	.021	.013	.003	.001			
8.0	.019	.011	.002				
9.0	.016	.007	.001				
10.0	.012	.005					

Integral Distribution $\int_{\varepsilon}^{\infty} P(\varepsilon)d\varepsilon$:

λ_w: Δ_w	1.48	1.45	1.4	1.2	1.0	0.8	0.5	0
-3.0								.999
-2.5								.994
-2.0	.998						.997	.977
-1.5	.990				.987		.969	.933
-1.0	.952				.932		.893	.841
-0.5	.849				.799		.760	.691
0.0	.720		.688	.666	.643	.617	.573	.500
0.5	.590		.547	.512	.463	.444	.385	.309
1.0	.484		.431	.381	.339	.300	.235	.159
1.5	.405		.342	.287	.237	.188	.130	.067
2.0	.341	.295	.273	.212	.158	.108	.067	.023
2.5	.292	.242	.228	.152	.101	.062	.032	.006
3.0	.256	.203	.175	.106	.064	.034	.015	.001
3.5	.225	.169	.139	.070	.041	.019	.006	
4.0	.202	.143	.113	.047	.026	.010	.002	
4.5	.181	.122	.092	.032	.016	.005	.001	
5.0	.163	.105	.076	.022	.010	.002		
5.5	.149	.089	.061	.015	.005	.001		
6.0	.137	.077	.048	.010	.004			
6.5	.126	.065	.038	.007	.002			
7.0	.117	.056	.031	.005	.001			
7.5	.109	.047	.025	.003				
8.0	.102	.039	.020	.002				
9.0	.090	.026	.015					
10.0	.081	.015	.009					

Table 6

Relations between Weighted and Unweighted Parameters

$\beta^2 = 0$

c	λ_w	λ_u	b	σ_u/ξ	σ_w/σ_u
12.2	1.45	1.75			
10.0	1.41	1.58	1.48	3.16	0.467
5.0	1.17	1.12	1.48	2.24	0.661
3.0	0.91	0.87	1.37	1.73	0.792
2.0	0.73	0.71	1.23	1.41	0.867
1.0	0.51	0.50	0.93	1.00	0.93
0.5	0.36	0.35	0.68	0.71	0.96

$\beta^2 = 1$

c	λ_w	λ_u	b	σ_u/ξ	σ_w/σ_u
15.6	1.35	1.86			
12.3	1.30	1.65			
10.0	1.24	1.49	1.32	2.23	0.59
7.7	1.15	1.31			
5.0	0.99	1.05	1.17	1.58	0.74
3.0	0.80	0.82	1.02	1.23	0.83
2.0	0.66	0.67	0.90	1.00	0.90
1.0	0.46	0.47	0.67	0.71	0.95
0.5	0.33	0.33	0.49	0.50	0.98

$\beta^2 = 0.4$

c	λ_w	λ_u
11.9	.40	1.77
8.1	1.30	1.46
4.7	1.10	1.11
3.0	0.91	0.88
2.0	0.73	0.73

Table 7

Moments of the Collision Spectrum vs W_k

W_k	$\frac{\chi_1}{2\mu_e}$	$\frac{\chi_2}{(2\mu_e)^2}$	$\frac{\chi_3}{(2\mu_e)^3}$
0.01000	414.2	1.0101	0.01012
0.01259	348.5	1.0127	0.01278
0.01585	292.8	1.0160	0.01615
0.01995	245.5	1.0202	0.02042
0.02512	205.7	1.0254	0.02587
0.03162	172.3	1.0321	0.03281
0.03981	144.3	1.0406	0.04170
0.05012	120.9	1.0514	0.05319
0.06310	101.61	1.0651	0.06791
0.07944	85.55	1.0826	0.08714
0.1000	72.30	1.1050	0.11235
0.1259	61.37	1.1338	0.1458
0.1585	52.40	1.1711	0.1906
0.1995	45.10	1.2194	0.2515
0.2512	39.17	1.2828	0.3360
0.3162	34.40	1.3162	0.4556
0.3981	30.60	1.4774	0.6293
0.5012	27.60	1.6278	0.8887
0.6310	25.28	1.8301	1.2894
0.7944	23.51	2.110	1.931
1.000	22.29	2.500	3.000
1.259	21.43	3.052	4.857
1.585	20.92	3.841	8.222
1.995	20.69	4.985	14.57
2.512	20.71	6.667	27.08
3.162	20.92	9.161	52.56
3.981	21.29	12.905	106.4
5.012	21.79	18.57	223.4
6.310	22.39	27.22	484.5
7.944	23.06	40.50	1079.7
10.00	23.80	61.00	2460.

q

W_k	I_z: 100 ev	200 ev	330 ev	600 ev	1000 ev
0.01	0.77	0.89	1.00	1.16	1.36
0.02	0.80				1.30
0.04	0.82				1.25
0.07	0.83				1.23
0.10	0.84	0.92	1.00	1.10	1.21
0.2	0.85				1.19
0.4	0.86				1.17
0.7	0.87				1.16
1.	0.87^5	0.94	1.00	1.08	1.15
2.	0.88^5				1.13^5
4.	0.89^5				1.12
7.	0.90^5				1.11
10.	0.91	0.96	1.00	1.07	1.10

Table 8

f_1, f_2, f_3, g_3 vs w_k

		f_1	
w_k	I_2: 330ev	100ev	1000ev
0.01000	0.00137	0.00105	0.00210
0.01585	0.00308	0.0024	0.0043
0.02512	0.00693	0.0055	0.0097
0.03981	0.01562	0.0126	0.021
0.06310	0.0352	0.029	0.044
0.1000	0.0791	0.065	0.098
0.1585	0.1759	0.147	0.21
0.2512	0.384	0.32	0.46
0.3981	0.816	0.69	0.96
0.6310	1.665	1.43	1.95
1.000	3.24	2.8	3.8
1.585	5.97	5.2	6.9
2.512	10.44	9.2	12.0
3.981	17.45	15.4	19.7
6.310	28.1	25.	32.
10.00	44.1	40.	49.

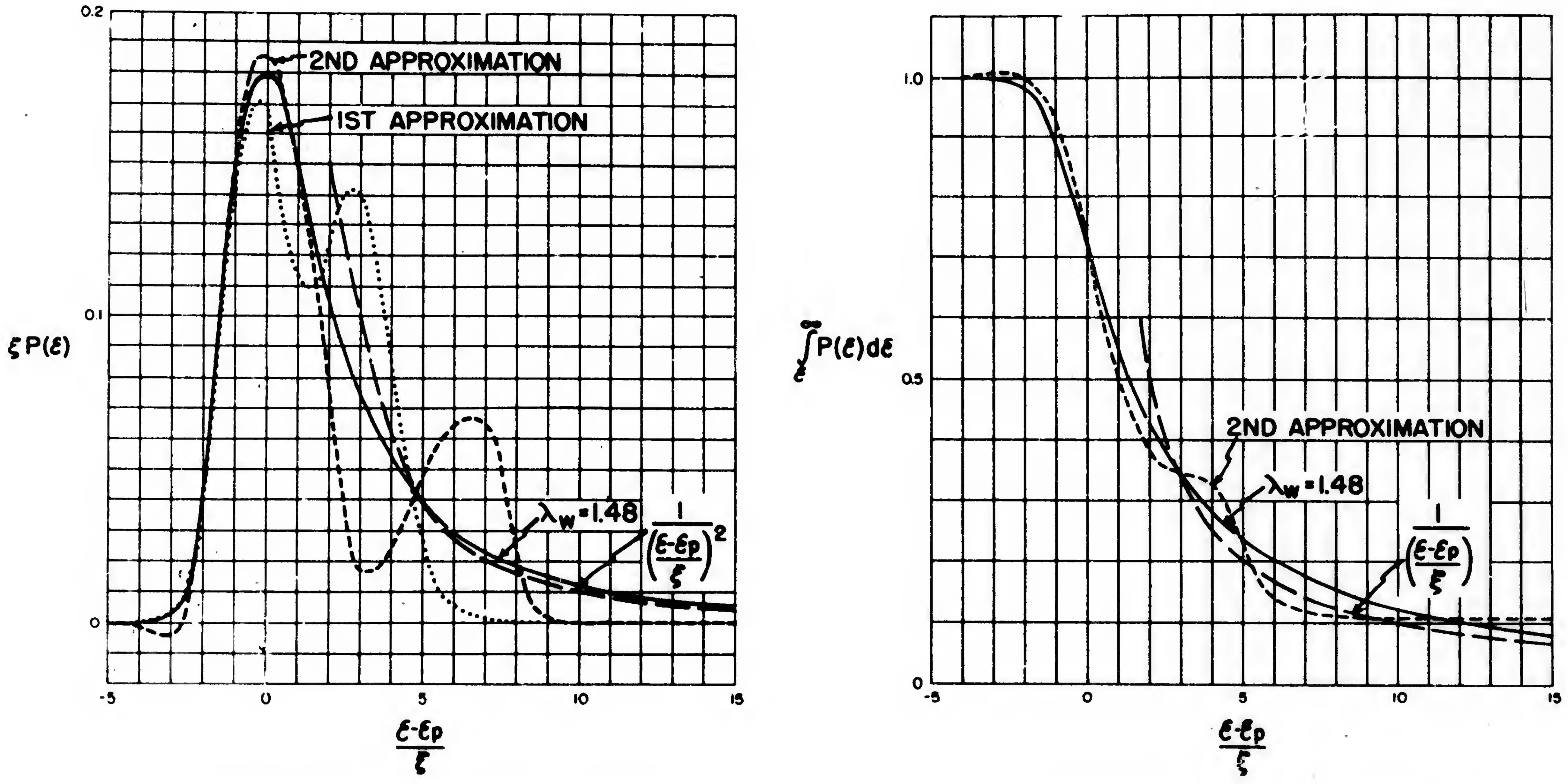
w_k	f_2	f_3	g_3
0.01000	4.21×10^{-9}	0.068×10^{-12}	0.068×10^{-6}
0.01585	19.38	0.694	0.221
0.02512	89.2	7.15	0.716
0.03981	414.	75.0	2.33
0.06310	$1925. \times 10^{-9}$	$788. \times 10^{-12}$	7.60×10^{-6}
0.1000	8.87×10^{-6}	0.00809×10^{-6}	0.0247×10^{-3}
0.1585	39.92	0.0799	0.0796
0.2512	172.1	0.736	0.251
0.3981	684.	6.10	0.759
0.6310	$2580. \times 10^{-6}$	47.3×10^{-6}	2.180×10^{-3}
1.000	8.82×10^{-3}	0.000284	0.00591
1.585	27.6	0.001593	0.01470
2.512	81.5	0.00840	0.0344
3.981	234.	0.0451	0.0770
6.310	$668. \times 10^{-3}$	0.253	0.1667
10.00	1.936	1.526	0.355

Table 9

Unit Collision Thickness for Various Materials

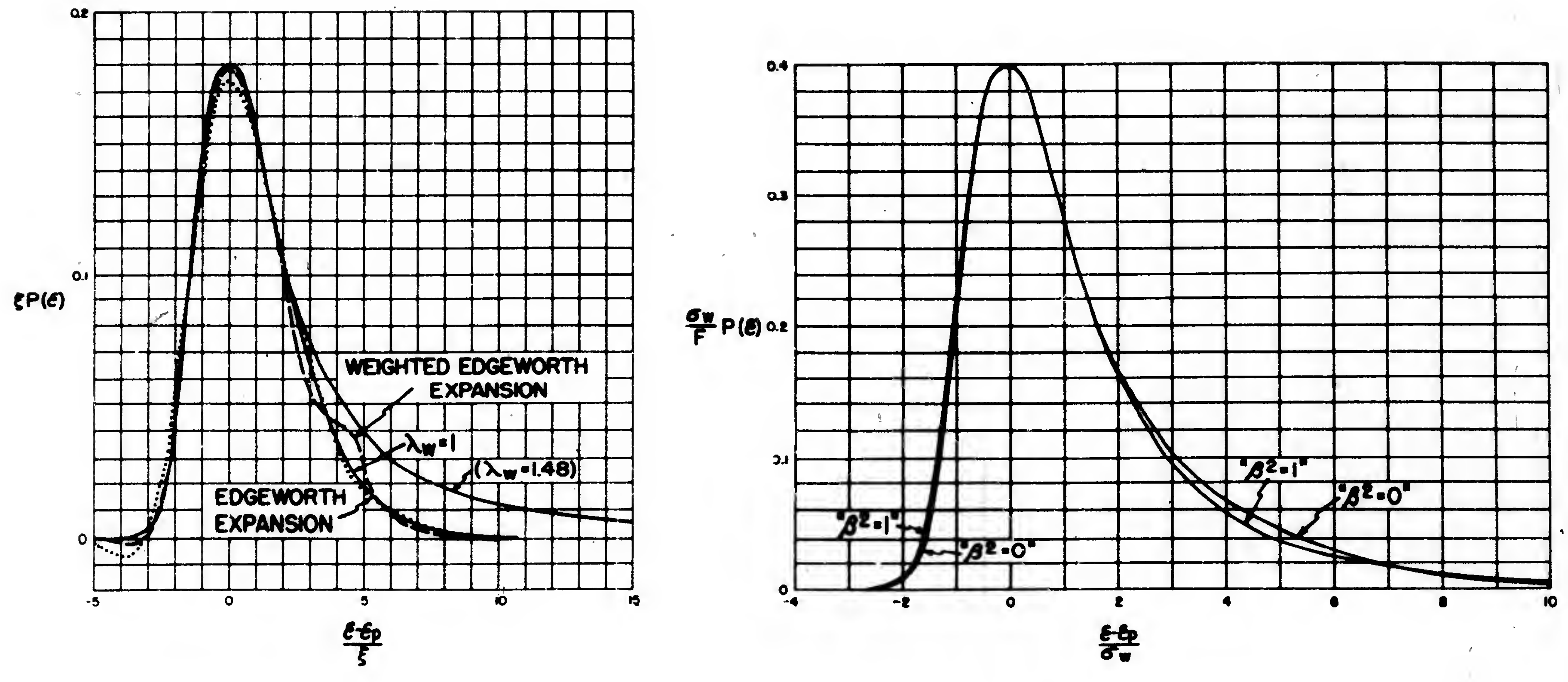
$$X = \pi\left(\frac{e^2}{mc^2}\right)^2 N \frac{Z}{A}\rho t = .150\,\frac{Z}{A}\rho t,$$ for ρ in gm/cm^3, t in cm

Material	$\rho t/x$ gm/cm^2	t/x cm
hydrogen	6.67	74,200 (STP)
air	13.3	1,030 (STP)
paraffin	11.6	13
water	12.0	12.0 (0°C)
aluminum	13.8	5.13
glass	13.4	5.1
Cement	13.4	4.8
silver chloride	14.9	2.69
iron	14.3	1.82
lead	16.8	1.48
gold	16.6	0.86



a. APPROXIMATE DIFFERENTIAL CURVES
$\lambda_w = 1.48$

b. APPROXIMATE INTEGRAL CURVES
$\lambda_w = 1.48$



c. APPROXIMATE DIFFERENTIAL CURVES
$\lambda_w = 1$

d. COMPARISON OF "$\beta^2 = 0$" AND "$\beta^2 = 1$" CURVES
$\lambda_w = 1.4$

Figure 1

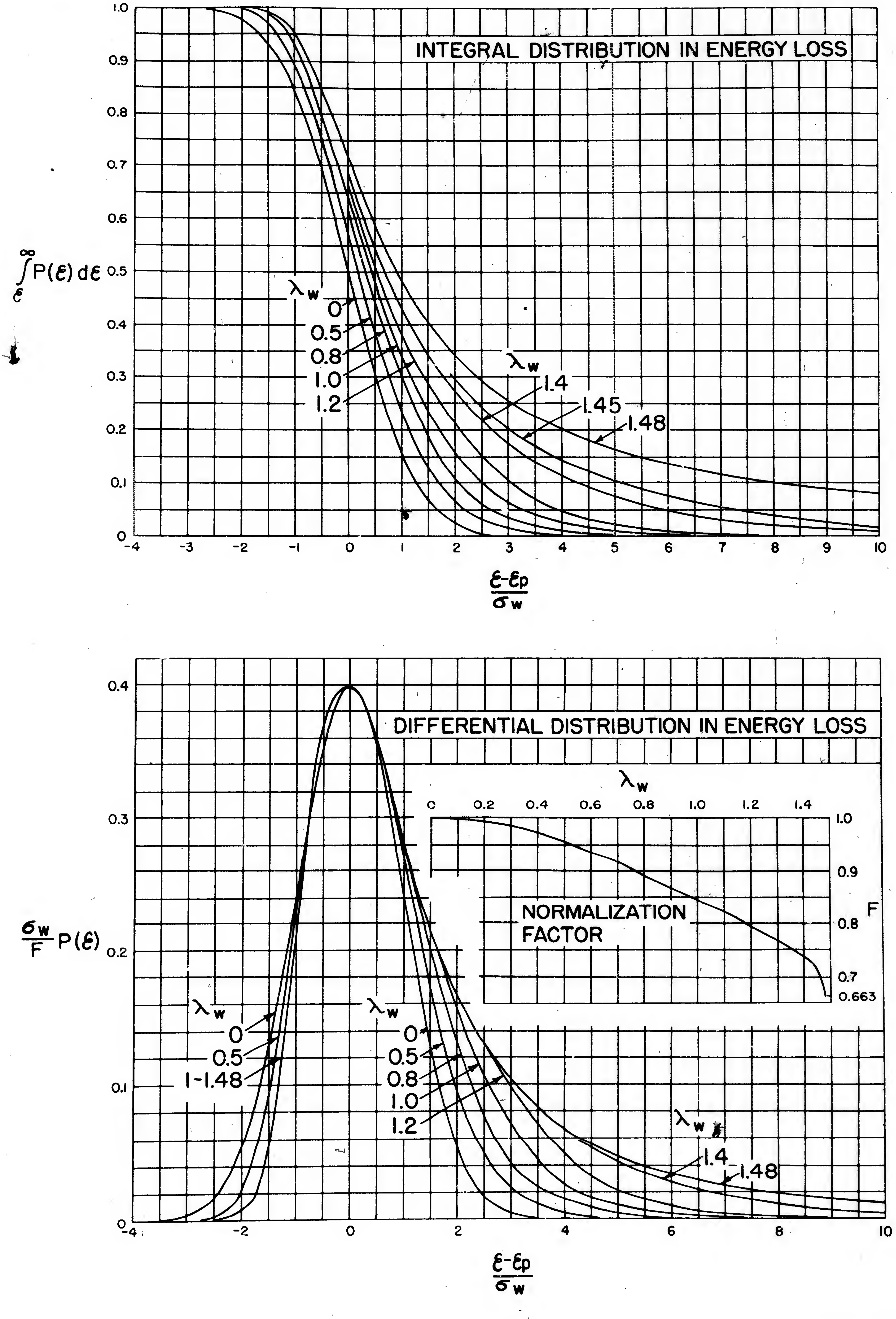
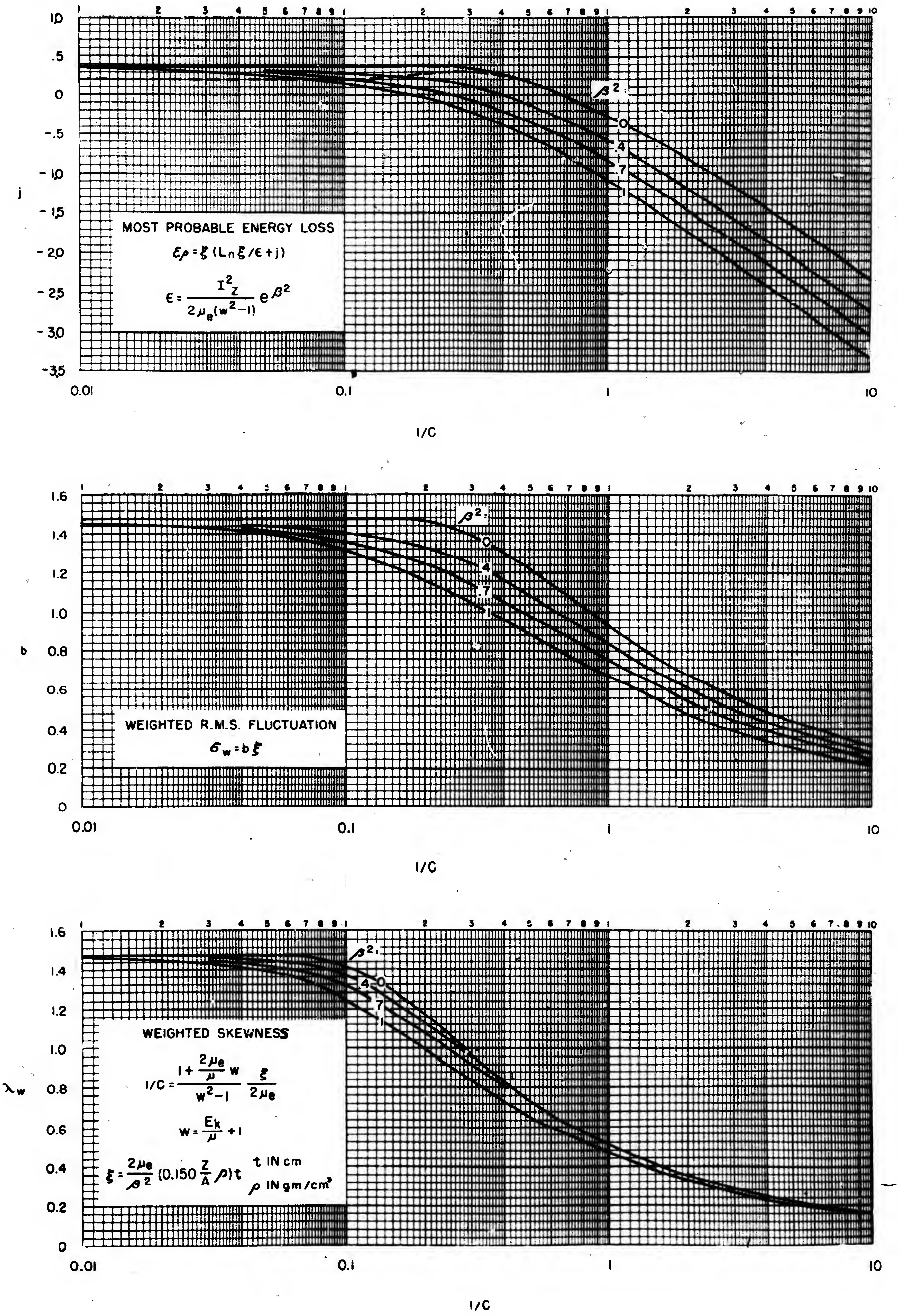
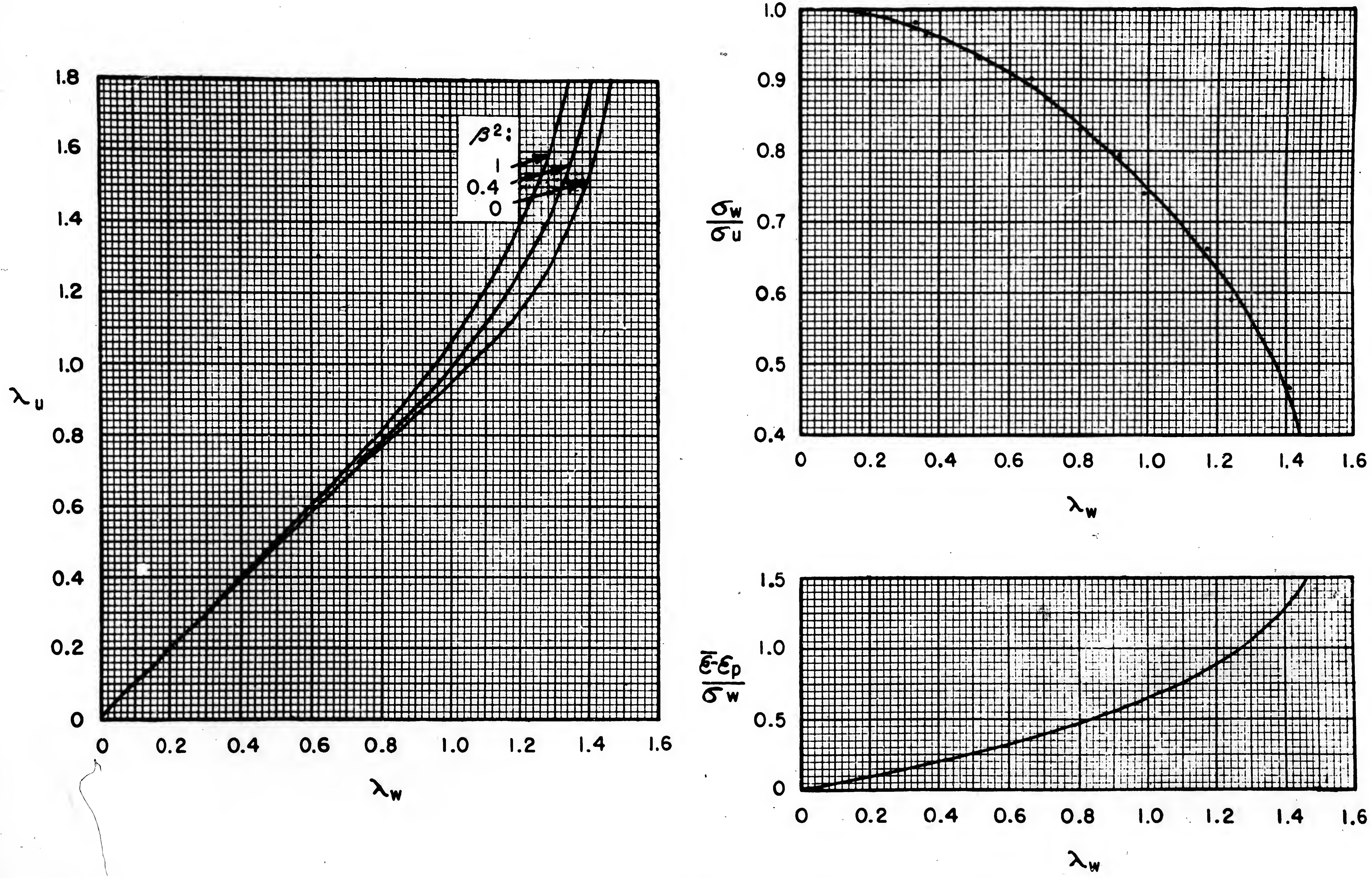

INTEGRAL DISTRIBUTION IN ENERGY LOSS
$\int_{\mathcal{E}}^{\infty} P(\mathcal{E})\, d\mathcal{E}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
$\frac{\mathcal{E}-\mathcal{E}_p}{\sigma_w}$
λ_w
0
0.5
0.8
1.0
1.2
λ_w
1.4
1.45
1.48
DIFFERENTIAL DISTRIBUTION IN ENERGY LOSS
$\frac{\sigma_w}{F} P(\mathcal{E})$
0.4
0.3
0.2
0.1
0
-4
-2
0
2
4
6
8
10
$\frac{\mathcal{E}-\mathcal{E}_p}{\sigma_w}$
λ_w
0
0.5
1-1.48
λ_w
0
0.5
0.8
1.0
1.2
λ_w
1.4
1.48
λ_w
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.0
0.9
0.8
0.7
0.663
F
NORMALIZATION FACTOR

Figure 2



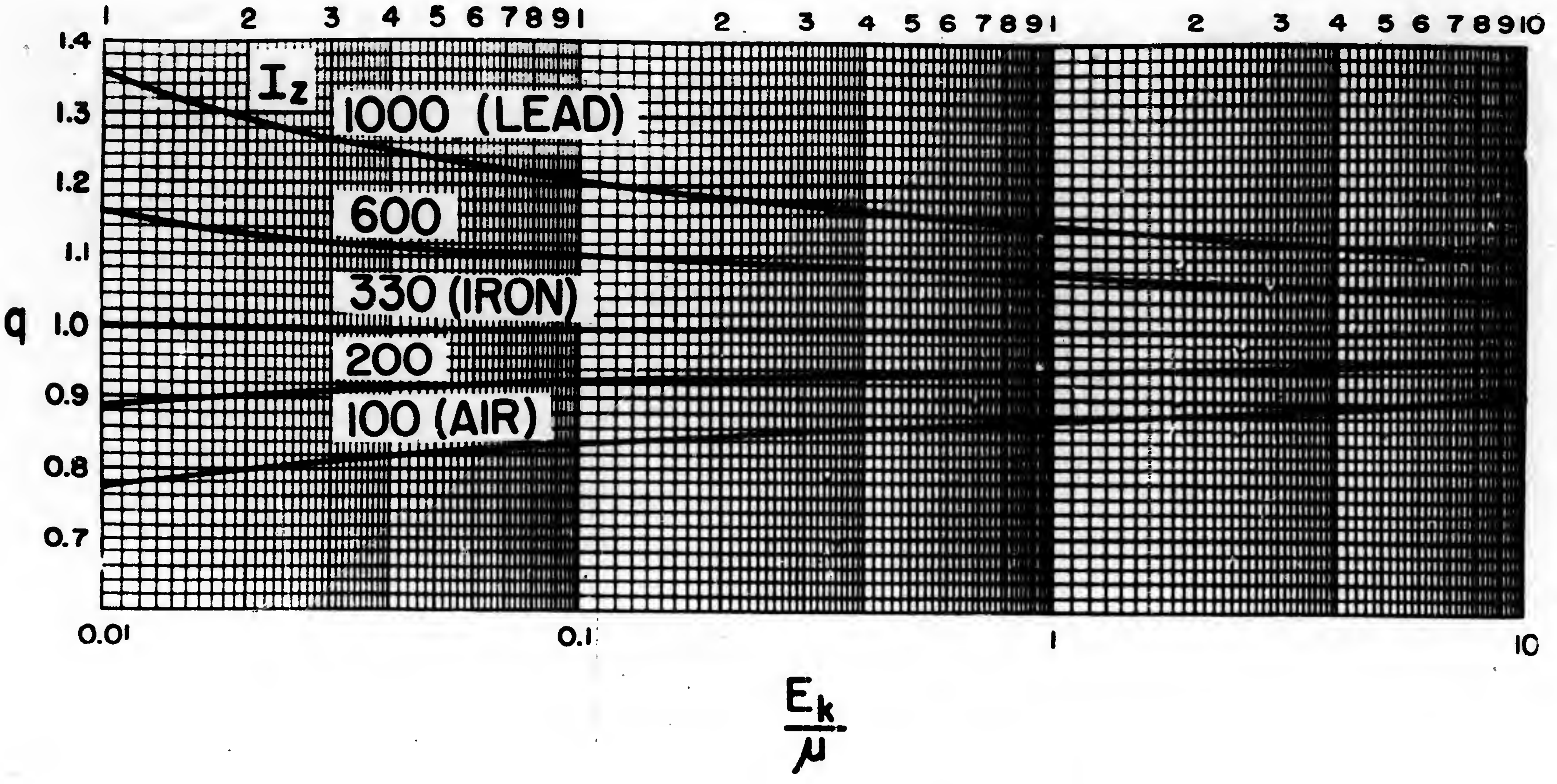
WEIGHTED PARAMETERS VS. THICKNESS OF MATERIAL TRAVERSED
(THESE CURVES MAY BE USED ONLY WHEN $\mathcal{E}_p$ IS A SMALL FRACTION OF E_k)

Figure 3

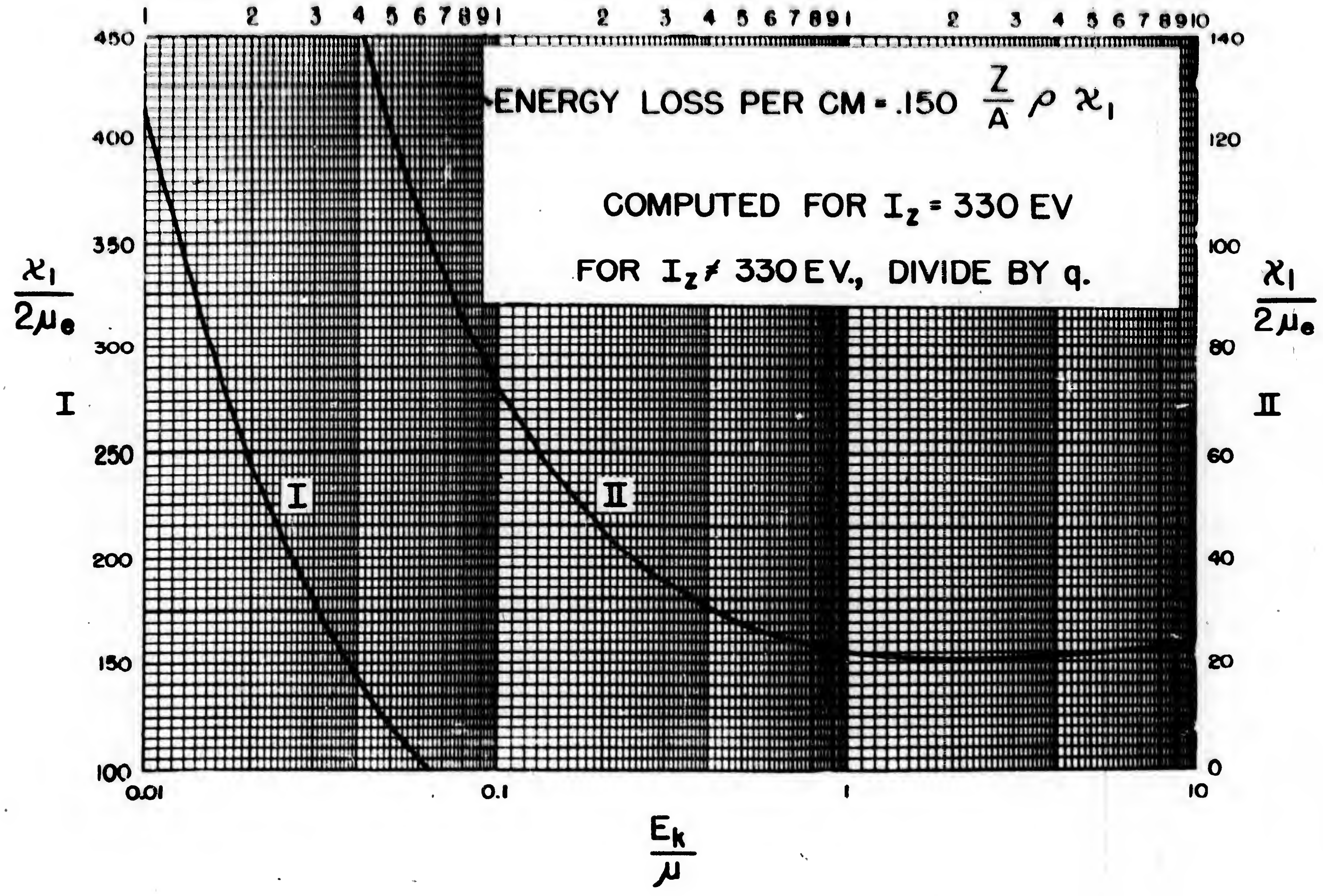


RELATIONS BETWEEN WEIGHTED AND UNWEIGHTED PARAMETERS

Figure 4



b. IONIZATION POTENTIAL CONVERSION FACTOR (q).



a. MEAN ENERGY LOSS PER CM. BY IONIZATION

Figure 5

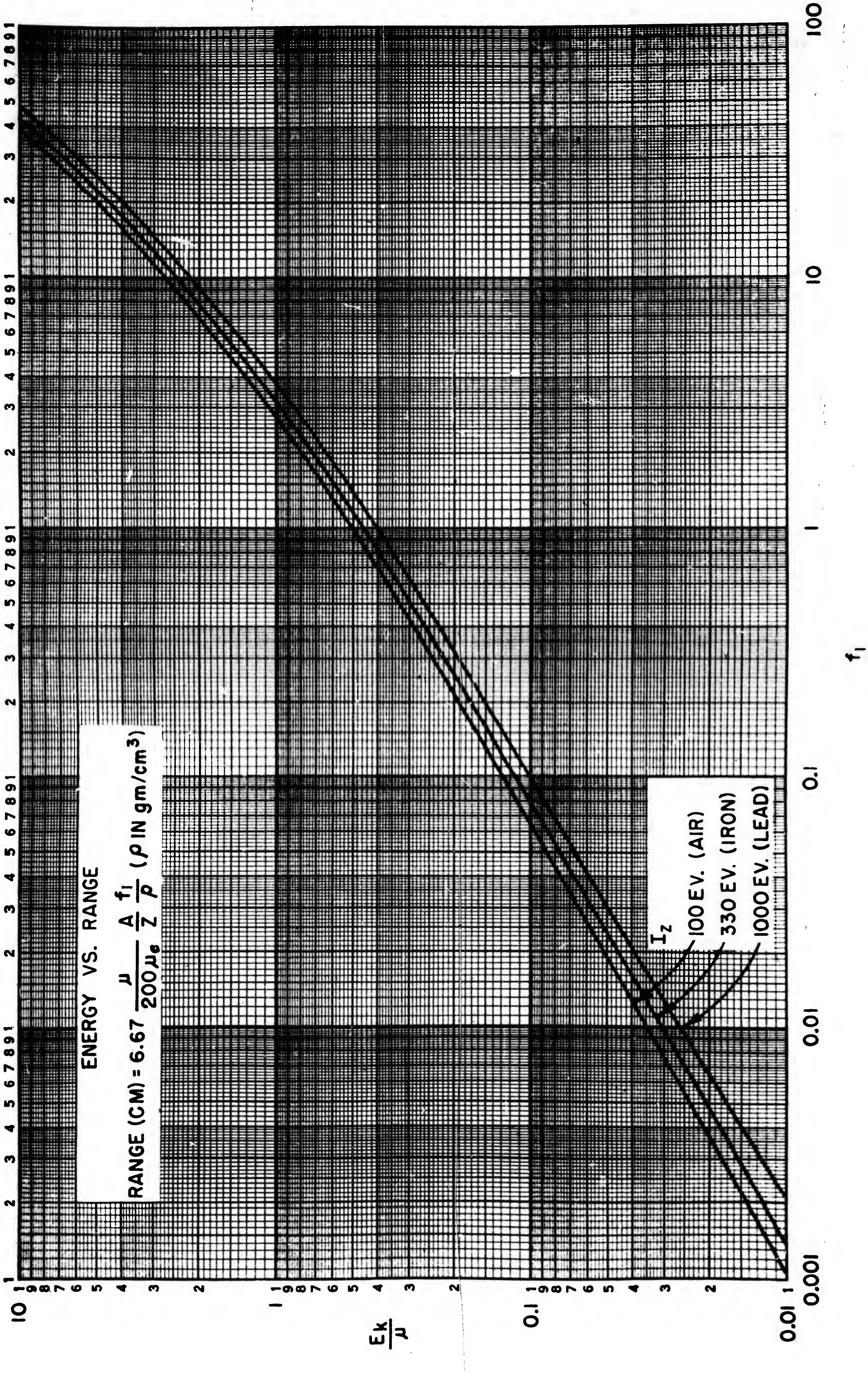

ENERGY VS. RANGE
RANGE (CM) = 6.67 $\frac{\mu}{200\mu_e}$ $\frac{A}{Z}$ $\frac{f_1}{\rho}$ (ρ IN gm/cm^3)
I_Z
100 EV. (AIR)
330 EV. (IRON)
1000 EV. (LEAD)
$\frac{E_k}{\mu}$
0.01
0.1
1
10
f_1
0.001
0.01
0.1
1
10
100

Figure 6

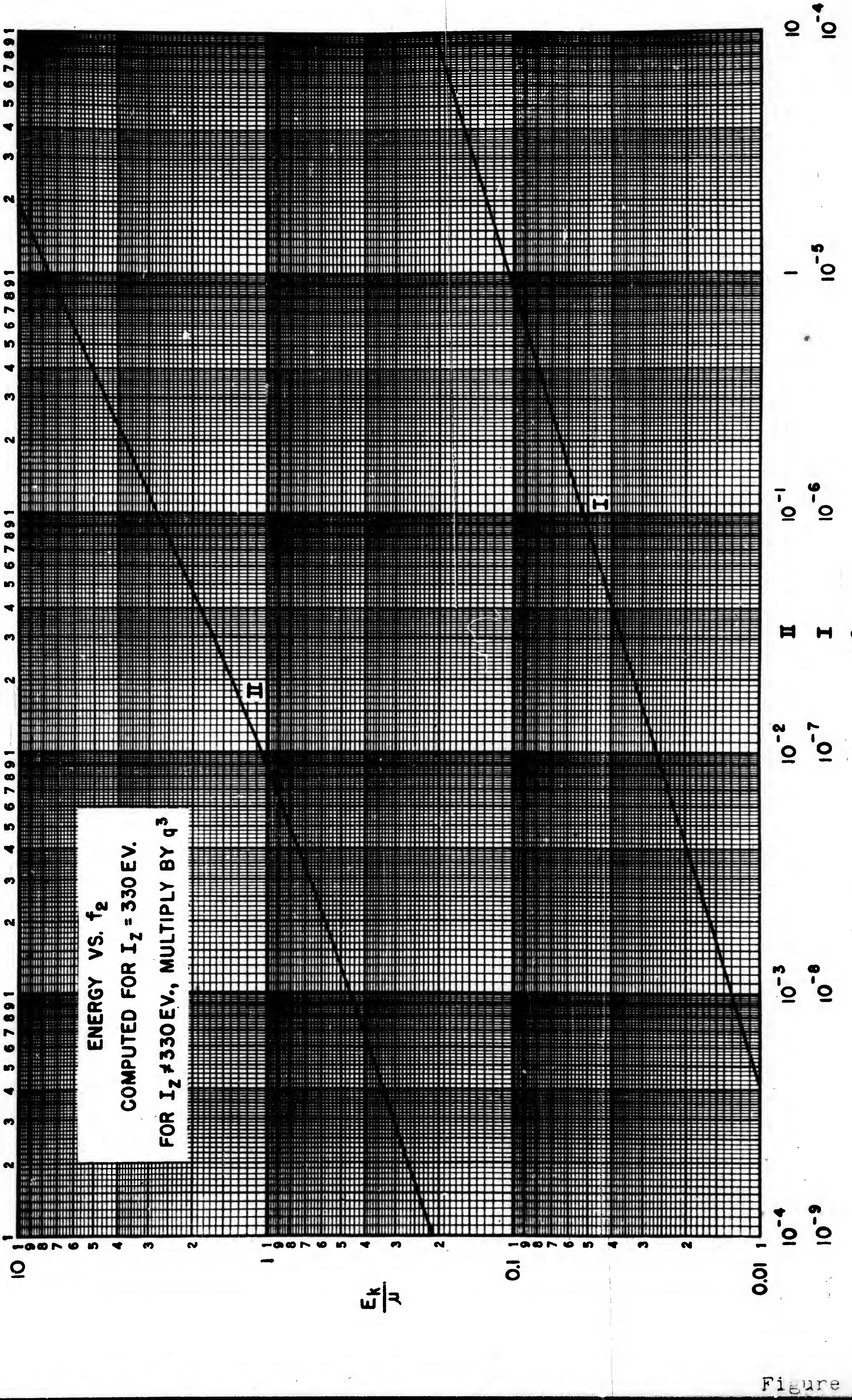

ENERGY VS. f2
COMPUTED FOR I_Z = 330 EV.
FOR I_Z ≠ 330 EV., MULTIPLY BY q^3
E_k/μ
10
1
0.1
0.01
II
I
f2
II: 10^-4, 10^-3, 10^-2, 10^-1, 1, 10
I: 10^-9, 10^-8, 10^-7, 10^-6, 10^-5, 10^-4

Figure 7

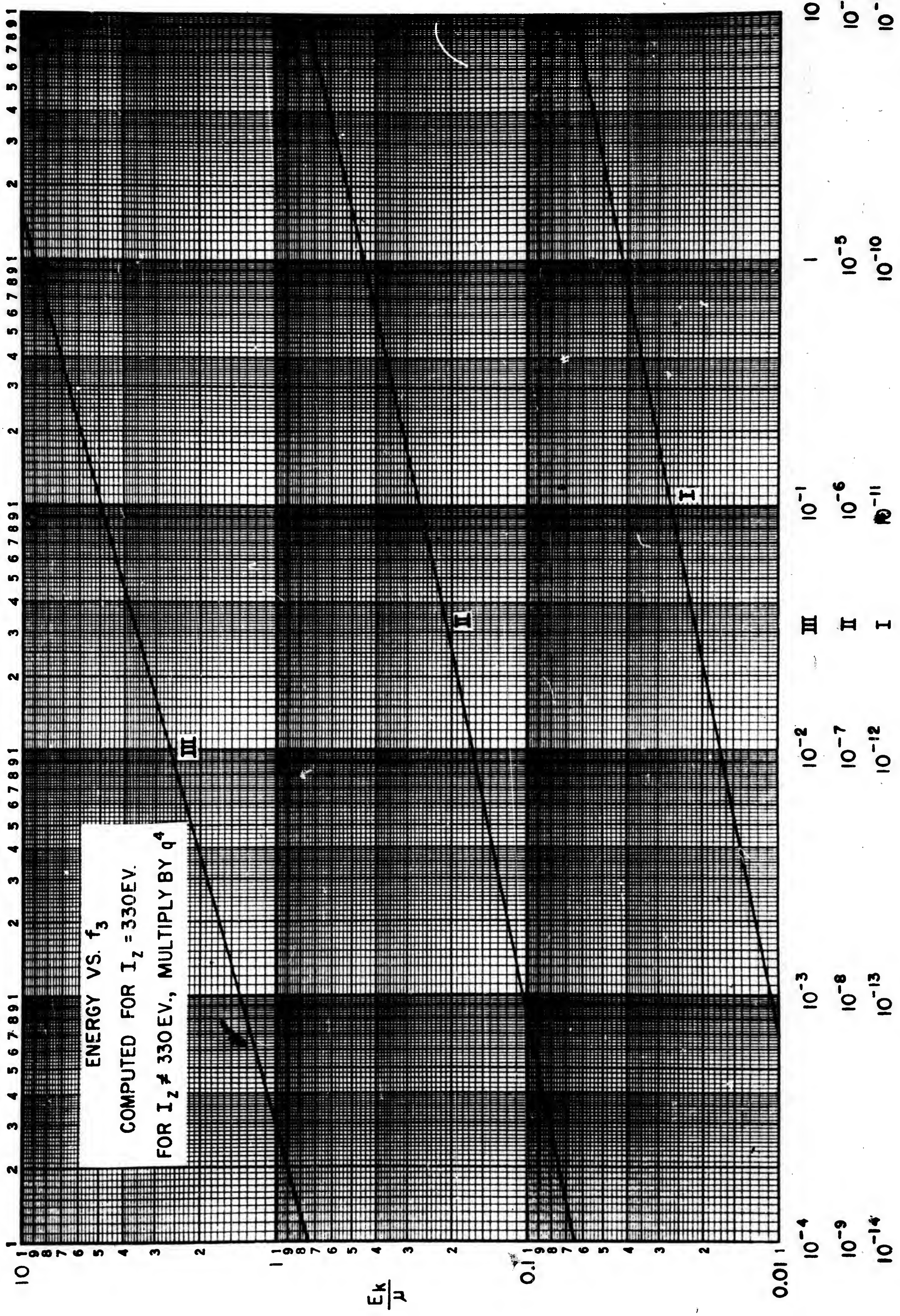

ENERGY VS. f_3
COMPUTED FOR I_Z = 330EV.
FOR $I_Z \neq$ 330 EV., MULTIPLY BY q^4
$\frac{E_k}{\mu}$
10
1
0.1
0.01
f_3
III
II
I
10^{-4} 10^{-3} 10^{-2} 10^{-1} 1 10
10^{-9} 10^{-8} 10^{-7} 10^{-6} 10^{-5} 10^{-4}
10^{-14} 10^{-13} 10^{-12} 10^{-11} 10^{-10} 10^{-9}

Figure 8

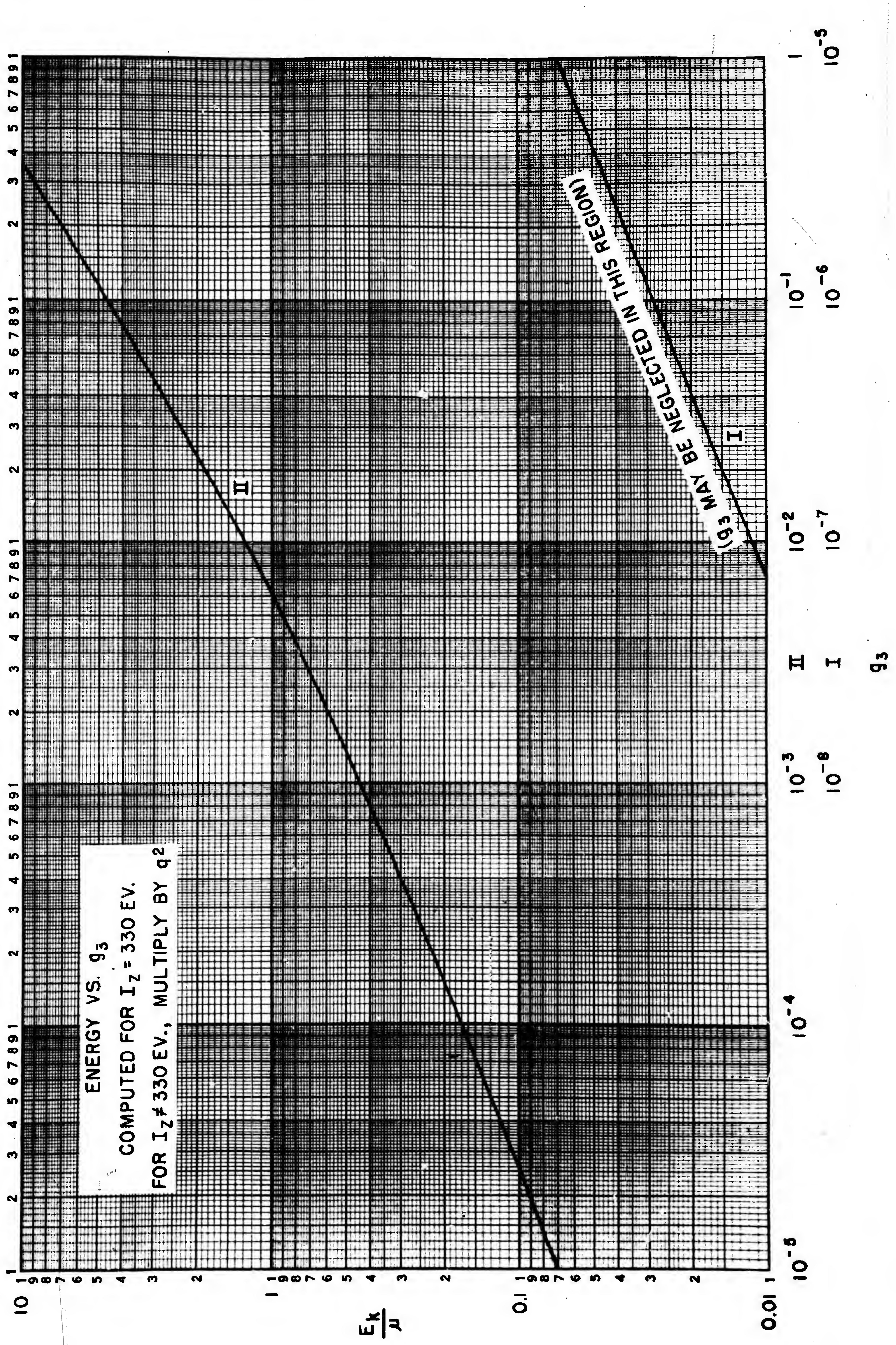
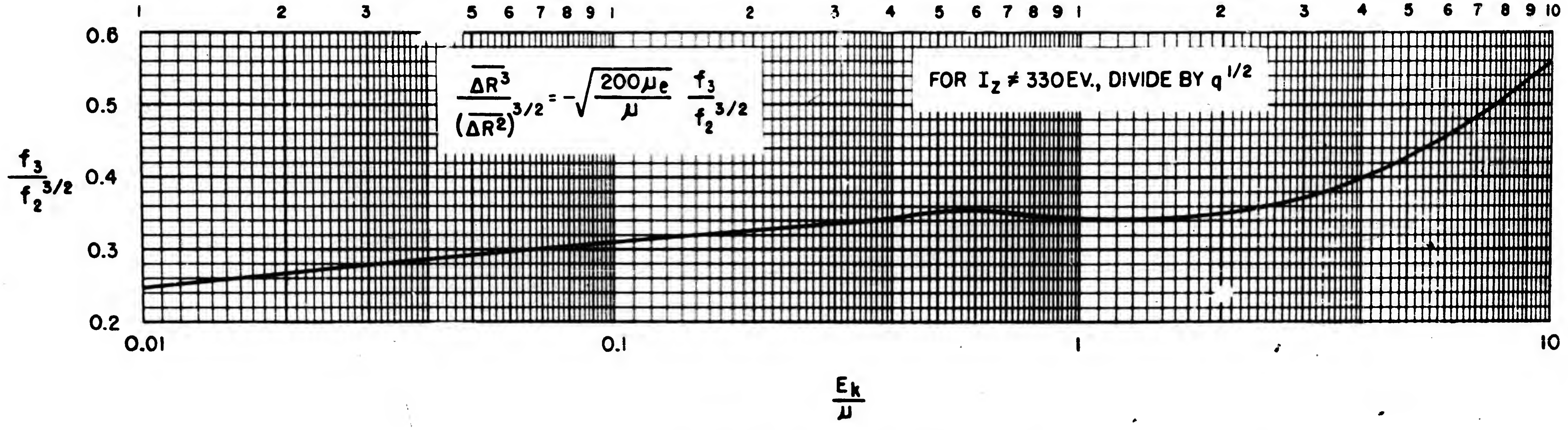
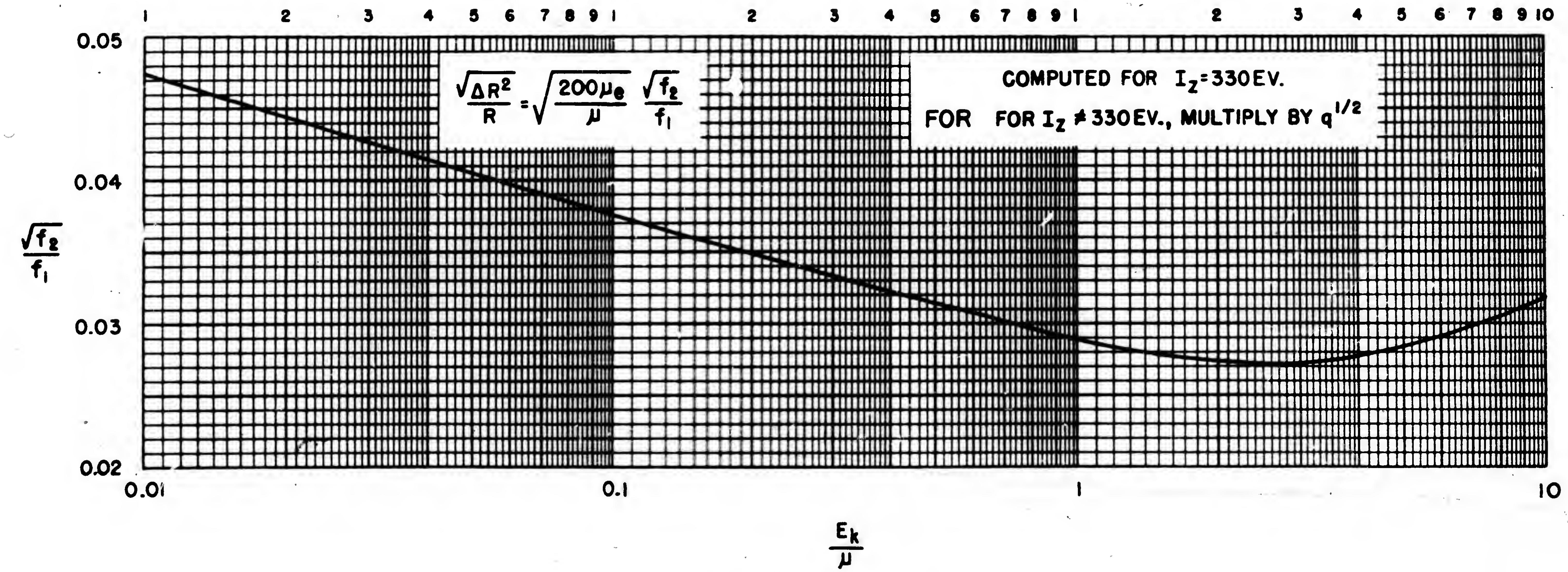

ENERGY VS. g_3
COMPUTED FOR $I_Z = 330$ EV.
FOR $I_Z \neq 330$ EV., MULTIPLY BY q^2
(g_3 MAY BE NEGLECTED IN THIS REGION)
II
I
$\frac{E_k}{\mu}$
10
1
0.1
0.01
g_3
II
I
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10^{-8}
10^{-7}
10^{-6}
10^{-5}

Figure 9



b. SKEWNESS IN RANGE DISTRIBUTION



a. FLUCTUATIONS IN RANGE

Figure 10

FLUCTUATIONS IN ENERGY LOST BY HIGH ENERGY CHARGED PARTICLES IN PASSING THROUGH MATTER

Abstract of a Thesis

Submitted in Partial Fulfillment of
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by

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ABSTRACT

The determination of the distribution in energy loss by particles traversing a given thickness of matter can be separated into two problems: (1) the determination of the cross sections for the collisions by which the particle may lose energy; (2) the calculation of the distribution in total energy loss resulting from such collisions. The first problem has been treated by many authors, and the cross sections given by the present relativistic quantum theory for collisions with atoms of electrons, positrons, protons, and mesons of spin 0, ½, or 1, may be regarded as known to a sufficient approximation. The second problem may be formulated as follows: given the probability $\chi(E')dE'dt$ that in going a distance dt, a particle will suffer a collision in which it loses an amount of energy between E' and $E'+dE'$, to calculate the probability $P(E,t)dE$ that after traversing a thickness t the particle has lost a total amount of energy between E and $E+dE$. The development of general methods of attacking this problem, and their application to the calculation of $P(E,t)$ for high energy mesons traversing various thicknesses of materials, constitute the central purpose of this thesis.

Methods of determining the distribution $P(E,t)$ are described which are applicable to problems of this type whenever the function $\chi(E')$ is known. The methods are particularly suited to cases when $\chi(E')$ is such that the probability that a particle will lose a large fraction of its kinetic energy in a single

collision is small, as is the case for collisions of heavy particles with electrons. These methods are applied to the determination of $P(\varepsilon,t)$ for mesons of cosmic ray energies. Curves are presented which can be used in the analysis of experiments involving meson energies from 1 to 1000 MEV, and any thicknesses of (homogeneous) material. The curves are also applicable to protons or other charged particles of mass large compared with the electron mass and of kinetic energies from 0.01 to 10 times the rest energy of the particles. Only energy losses due to collisions with electrons have been taken into account in the calculations.

The problem of the effect of scattering by collisions on the spatial distribution and the distribution in direction of motion is not treated in this thesis. The angular deflections suffered by mesons are very small except at very low energies and except for rare close collisions with an atomic nucleus. When directional scattering is important, the distribution $P(\varepsilon,t)$ can still be determined by the methods described, but the variable t must be interpreted as the actual path length of the particle in the material.

The problem of determining the distribution in energy loss of fast charged particles was probably first attacked by Bohr in 1915. Bohr's treatment was extended and improved in 1929 by Williams, who has written a number of papers comparing the experimental and theoretical results for the stopping of α- and β-particles. In his later papers, quantum and relativistic corrections are made, and it is concluded that theory and experiment agree within the expected limits of error. In a recent paper, Landau has given a rigorous solution of the problem of determining the energy loss distribution for high energy particles in the limiting case of moderately thin foils. In the present thesis, an attempt is made to determine the distribution $P(\epsilon,t)$ for larger thicknesses t when Landau's approximations are not valid and when the dependence of $\chi(E')$ on the energy of the incident particle must be taken into account.

The thesis begins with a short history of the problem, followed by a sketch of the methods by which cross sections for meson electron collisions are computed using relativistic quantum theory. The formulas for mesons of spin 0 are used in the later calculations, The results will be applicable to mesons of any spin within the range of energies treated. A brief discussion is included of the importance of other modes of energy loss (collision with nuclei,radiation) and their effect on the accuracy of the results.

The fundamental integro-differential equation governing the probability function $P(\epsilon,t)$ is derived for the general case in which the function $\chi(E')$ may depend on the thickness t of material traversed, and on the energy E of the incident particle. The treatment of this equation falls into two parts, according to whether or not the dependence of $\chi(E')$ on the energy E of the incident particle is taken into account. The general solution for $P(\varepsilon,t)$ is derived in terms of a Laplace transform, for the case where $\chi(E')$ is independent of ε but may depend on the depth t in the material. The special case when Landau's assumptions are valid (t small) is discussed. The opposite case when t is large and the distribution in energy loss is nearly Gaussian can be treated by means of an Edgeworth, or Gram-Charlier expansion. The general relations are derived between the moments of the function $\chi(E')$ and the semi-invariants and moments of the energy loss distribution $P(\varepsilon,t)$. A modification of the Edgeworth expansion is presented, based on the connection between the Edgeworth expansion and the saddle-point method, which gives a series asymptotically valid for large values of t, but also useful when t is small. The modified Edgeworth expansion leads naturally to the introduction of a set of weighted moments which are better suited to characterize the distribution curves for small values of t than the ordinary unweighted moments. Other methods are outlined which are often useful in determining $P(\varepsilon,t)$.

All these methods, together with the results of Landau, are finally combined in the calculation of a family of distribution curves $P(\varepsilon,t)$ and $\int_{\varepsilon}^{\infty} P(\varepsilon,t)\,d\varepsilon$. It is shown that, to a sufficient degree of approximation, any $P(\varepsilon,t)$ can be represented by one of the curves of this family if the three "weighted parameters" which characterize it are properly chosen. This result holds whenever $P(\epsilon,t)$ is a solution corresponding to a function $\chi(E')$ of the form derived for incident scalar mesons, and when the dependence of $\chi(E')$ on the energy ϵ lost by the meson is neglected. Curves are given for the three weighted parameters, as functions of the initial energy and thickness, and as functions of the unweighted moments of the distribution $P(\epsilon,t)$.

If we make the plausible assumption that when the variation of $\chi(E')$ with E is taken into account, the curves mentioned in the last paragraph may still be used provided the parameters are properly chosen, it remains only to determine the parameters in order to have a solution for all thicknesses t. Equations for the unweighted moments of $P(\epsilon,t)$ are derived, which are valid when $\chi(E')$ is a function also of E. Curves are given from which the three required parameters may be determined for thicknesses t up to about 90% of the range for mesons.

When the distribution $P(\epsilon,t)$ is known, other related distributions can be determined, such as the distribution in distances traversed in losing a given amount of energy, in particular, the distribution in range of particles of given initial energy; and the probability distribution of the initial energy, given the depth and final energy, or given the range. Formulas for these distributions are derived and curves are given.

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